On the Dimensions of Linear Spaces of Real Matrices of Fixed Rank

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Abstract

This thesis studies the problem of estimating the largest possible dimension of a linear space of real matrices under the assumption that every non-zero matrix in the space has (the same) fixed rank.

The complex version of this problem has been studied by R. Westwick and J. Sylvester. Sylvester introduced a technique based on the theory of Chern classes for estimating the dimension from above.

The question of determining the largest dimension of a linear space of maximal-rank real $n \times n$ matrices (or, equivalently, of determining the largest number of nonsingular $n \times n$ matrices all of whose non-trivial linear combinations are nonsingular) was solved by J.F. Adams, P. Lax and R. Phillips. Their proof uses Adams' solution of the vector fields on spheres problem to show that the linear spaces constructed by J. Radon and A. Hurwitz are of the largest possible dimension under this hypothesis.

A number of general results on the dimensions of real linear spaces of fixed-rank singular matrices, are due to R. Meshulam, E. Rees, K.Y. Lam and P. Yiu. The method used to provide upper bounds for the dimension is analogous to the complex case; here Stiefel-Whitney classes and K-theory are used for the calculations. Clifford Algebras are then used to construct spaces and so provide lower bounds for the dimension. For spaces of large rank the maximum dimension is related to the Radon-Hurwitz function, $\rho(n)$. In particular, for spaces of $n \times n$ matrices of rank $n-k$ with $n \geq 8$ and $k = 1, 2$, the maximum dimension is given by the largest of the $2k+1$ integers $\rho(n-k), \ldots, \rho(n), \ldots, \rho(n+k)$.

We show that for $k = 3$ or 4 this relation continues to hold for almost all values of n. Sufficient conditions for the remaining cases are formulated in terms of lower bounds for the geometric dimensions of certain sums of line bundles over real projective spaces.

Also considered are similar questions for spaces of rectangular matrices. We show how calculations with Stiefel-Whitney classes together with information about the existence of certain bilinear maps enable us to determine the dimensions of spaces of real $n \times k$ matrices of fixed-rank k for all n and k with $k \leq 9$.

The case of fixed-rank symmetric matrices is also investigated. The main result here is that every space of real symmetric $n \times n$ matrices of fixed rank $2k+1$ must have dimension 1.

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Introduction

The theory of matrices plays a central role in mathematics. Although it may be thought of in the purely abstract setting of linear algebra, many examples arise naturally in such diverse areas as differential equations, statistics and number theory. Indeed, the theory has applications to almost every branch of mathematics.

Many questions concerning vector spaces of matrices, it seems, cannot be answered by using linear algebra alone. For example, problems concerning spaces of matrices of bounded rank lead to the consideration of certain varieties in algebraic geometry. This situation has been studied in [F],[HT],[R4],[R5], and elsewhere.

We will be concerned with estimating the largest possible dimension of those linear spaces of matrices for which every non-zero matrix in the space has the same rank. For example, consider the 3-dimensional space

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

Every non-zero matrix in this space has rank 2; that there can be no space of larger dimension follows from a result of Meshulam [M1].

An account of the history and solution of the problem for nonsingular matrices is presented in chapter 1. The main result, that the spaces spanned by the orthogonal matrices constructed by Radon [R1] and Hurwitz [H1] are of the largest possible dimension, is a consequence of Adams' celebrated solution of the vector fields on spheres problem [A1].

Adams, Lax and Phillips [ALP],[ALP2] combined this result with some elementary constructions to answer the equivalent questions for spaces of nonsingular complex and quaternionic matrices. Their results include calculations of the largest dimensions of spaces of real symmetric matrices and of complex and quaternionic Hermitian matrices. A summary of their methods and results is given in section 2 of chapter 1.

The second chapter presents the known results on the general case, in which spaces of $m \times n$ matrices of fixed-rank k are considered. Some results for spaces of complex matrices are proved by Westwick [W1],[W2],[W3],[W4] and Sylvester [S2]. In particular, Sylvester showed that Chern classes can be used to estimate the maximum dimension from above.

A similar approach has been adopted by Meshulam [M1], and more recently by Rees [R2],[R3], Lam and Yiu [LY2] to study spaces of real matrices; here calculations with Stiefel-Whitney classes and K-theory are used instead of Chern classes. The main idea, which is described in detail in chapter 2 and forms the basis for further results in later chapters, is the following:

An $m \times n$ matrix A with entries in R (say) is a linear transformation $A : R^n \longrightarrow R^m$, and induces a short exact sequence of vector spaces

$$0 \longrightarrow \mathrm{Ker}(A) \longrightarrow R^n \xrightarrow{A} R^m \longrightarrow R^m/\mathrm{Im}(A) \longrightarrow 0.$$

The rank of a matrix measures the surjectivity of the corresponding map. Moreover, if V is a $(d+1)$-dimensional linear space of rank k matrices then a short exact sequence of algebraic vector bundles over the real projective space $P(V)$ can be constructed; if one can show that such an exact sequence cannot exist, then d is an upper bound for the dimension of all such linear spaces.

The maximum dimension of spaces of $n \times n$ matrices for which the common rank k is very large is closely related to the Radon-Hurwitz function, $\rho(n)$: if the positive integer n is written in the form $(2a+1)2^b$ with $b = c + 4d$ and $0 \leq c \leq 3$, then the function is defined by

$$\rho(n) = 2^c + 8d.$$

Clifford algebras are a useful tool for constructing spaces of matrices of large rank, and thus of obtaining lower bounds for dimension. A combination of these methods is used in [R2] and [LY2] to show that for $k = n-1$ or $n-2$ the maximum dimension is equal to the function $\rho(n, n-k)$, defined by

$$\rho(n, n-k) = \max(\rho(n-k), \dots, \rho(n), \dots, \rho(n+k)),$$

providing $n \geq 8$. Some low-dimensional cases give different results.

The relationship between the maximum dimensions of certain large-rank spaces and the function $\rho(n, n-k)$ is investigated in more detail in chapter 3. For $n \times n$ matrices of rank $n-3$ or $n-4$, it turns out that in almost all cases equality can be established. It is conjectured that the relation holds for all n sufficiently large. The conjecture for the outstanding cases can be formulated in terms of lower bounds for the geometric dimension of certain sums of tautological line bundles over RP^n. Unfortunately, the method described above does not apply to these exceptional cases. However, in trying to obtain a proof, a small improvement on the known results on geometric dimension was made. (The proof is outlined in the appendix.)

As the common rank k decreases, so does the number of cases for which the relation with $\rho(n, n-k)$ can be proved (by these methods). In such cases, though, we can often establish a weaker dependence of the maximum dimension upon the largest power of 2 dividing n (i.e. on b, in the above notation).

A generalization of the 'large-rank' problem, whereby one considers the largest possible dimensions of spaces of $m \times n$ matrices where m and n are not necessarily equal, is closely related to questions concerning the existence of certain nonsingular bilinear maps. Much of the literature on this is due to Lam (see [L1], [L2], for example). In fact, for the case of spaces of rectangular matrices of maximal rank, the two notions coincide.

Chapter 4 extends the known results on low-rank spaces. The maximum dimension of all $m \times n$ matrices of fixed rank k is determined for $2 \leq k \leq 4$, and partial results for $5 \leq k \leq 9$ are given. A technique illustrating the essentially algorithmic nature of the calculations for some low-dimensional examples is presented. The method also illustrates how calculations with Stiefel-Whitney classes can sometimes give stronger information than that obtained using K-theory. The chapter concludes with a table giving the largest dimensions of all spaces of $n \times n$ matrices of rank k for $k \leq n \leq 12$.

In the final chapter, spaces of real symmetric matrices are considered. The methods used differ from those in earlier chapters. In particular, imposing the condition that every non-zero matrix in a linear space of real symmetric matrices has fixed rank gives rise to interesting restrictions on the way the eigenvalues of matrices in such a space vary over the space. This is used to show that every symmetric space of odd rank must be one-dimensional. The argument can also be adapted to simplify the calculations for some spaces of even rank.

Chapter 1

Nonsingular Spaces

1.1 Real spaces

Consider a system $A_1, \ldots, A_r$ of orthogonal real $n \times n$ matrices satisfying the matrix equations

$$\begin{aligned} A_i A_i^t &= I_n & 1 \le i \le r; \\ A_i A_j^t + A_j A_i^t &= 0 & 1 \le i, j \le r, \ i \ne j. \end{aligned}$$

Such a system is equivalent to a 'sums of squares formula'

$$(x_1^2 + \ldots + x_r^2)(y_1^2 + \ldots + y_n^2) = z_1^2 + \ldots + z_n^2$$

where the z_i are bilinear forms in the x_i, y_j with real coefficients.

The system is also equivalent to a normed bilinear map $f : R^r \times R^n \longrightarrow R^n$ satisfying

$$|f(x,y)| = |x||y| \qquad \forall x \in R^r, \ y \in R^n.$$

A natural question to ask is, for a given n, what is the largest integer r for which an orthogonal system (or sums of squares formula) exists?

Remark The problem can be generalized to systems of $m \times n$ matrices, though the matrix equations become much harder to handle. There are applications to many areas of mathematics. A good survey article on sums of squares formulae is [T2].

Note that the problem can be stated more invariantly: an orthogonal system $A_1, \ldots, A_r$ of $n \times n$ real matrices is a basis for a r-dimensional vector space with the property that every non-zero matrix in the space is nonsingular. To see this, consider an arbitrary non-trivial linear combination $B = \lambda_1 A_1 + \ldots + \lambda_r A_r$ of the A_i. Then

$$BB^t = (\lambda_1 A_1 + \ldots + \lambda_r A_r)(\lambda_1 A_1^t + \ldots + \lambda_r A_r^t).$$

By the orthogonality properties, this equals

$$(\lambda_1^2 + \ldots + \lambda_r^2) I_n,$$

which is nonsingular since the λ_i are not all zero. So $(\det(B))^2 = \det(BB^t) \neq 0$ and B is nonsingular.

1.1.1 Radon-Hurwitz matrices

Systems of orthogonal square matrices were studied in the early 1920s by Radon [R1] and Hurwitz [H1]. The following function plays a central role.

Definition 1.1.1 *For a positive integer n, write $n = (2a+1)2^b$, where $b = c + 4d$ and $0 \leq c \leq 3$. The Radon-Hurwitz number $\rho(n)$ is defined by $\rho(n) = 2^c + 8d$.*

Theorem 1.1.2 (Radon and Hurwitz) *For each positive integer n, there exists a system of r orthogonal matrices if and only if $r \leq \rho(n)$.*

Note Eckmann [E] gave an alternative proof of this theorem in 1942.

The following summary of the construction of orthogonal systems follows [LY1] (see also [R3]).

Identify R^2 with the complex numbers C, R^4 with the quaternions H and R^8 with the Cayley numbers K. For $b = 1, 2$ and 3 take the standard orthonormal bases $e_0 = 1, e_1, \ldots, e_{2^b-1}$ for C, H, and K. The bases satisfy

$$\begin{aligned} e_i^2 &= -1 && (1 \leq i \leq 2^b - 1) \\ e_i e_j &= -e_j e_i && (1 \leq i, j \leq 2^b - 1,\ i \neq j). \end{aligned}$$

Define matrices $E_{b,j}$ for $0 \leq j \leq 2^b - 1$ by left multiplication by the basis element e_j. The $E_{b,j}$ form a system of orthogonal matrices of order $2^b = \rho(2^b)$.

Example: the quaternions. The matrices $E_{2,0}, E_{2,1}, E_{2,2}$ and $E_{2,3}$, corresponding to left multiplication by $1, i, j$ and k are respectively given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The real linear combinations $aE_{2,0} + bE_{2,1} + cE_{2,2} + dE_{2,3}$ give a 4-dimensional space

$$\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}.$$

Example: the Cayley numbers. The 8-dimensional space obtained is

$$\begin{bmatrix} a & -b & -c & -d & -e & -f & -g & -h \\ b & a & -d & c & -f & e & h & -g \\ c & d & a & -b & -g & -h & e & f \\ d & -c & b & a & -h & g & -f & e \\ e & f & g & h & a & -b & -c & -d \\ f & -e & h & -g & b & a & d & -c \\ g & -h & -e & f & c & -d & a & b \\ h & g & -f & -e & d & c & -b & a \end{bmatrix}.$$

For $n = 16$, define 9 $(= \rho(16))$ orthogonal matrices as follows. Regard Cayley multiplication as a map $R^8 \times R^8 \longrightarrow R^8$ and consider the map

$$R^{16} \times R^{16} \longrightarrow R^{16}; \qquad (a, b).(c, d) \mapsto (ac - \bar{d}b, da + b\bar{c}).$$

If a is any Cayley number and b is real then the map is singular only if $(a, b) = (0, 0)$. Denote the 9 linearly independent maps by $I_{16}, \gamma_1, \ldots, \gamma_8$. They are orthogonal transformations, and satisfy the relations

$$\gamma_i^2 = -I_{16} \ (1 \leq i \leq 8) \qquad \text{and} \qquad \gamma_i\gamma_j = -\gamma_j\gamma_i \ (i \neq j).$$

Now assume by induction that there are $\rho(n)$ orthogonal $n \times n$ matrices $I_n, A_2, \ldots, A_{\rho(n)}$ satisfying $A_i^2 = -I_n$ $(2 \leq i \leq \rho(n))$ and $A_iA_j = -A_jA_i$ $(i \neq j)$. Let $B = \gamma_1\gamma_2 \ldots \gamma_8$. Then the $16n \times 16n$ matrices (regarded as transformations of $R^n \otimes R^{16}$) given by

$$I_n \otimes I_{16}, A_2 \otimes B, \ldots, A_{\rho(n)} \otimes B, I_n \otimes \gamma_1, \ldots, I_n \otimes \gamma_8.$$

are orthogonal (and satisfy analogous relations). There are $\rho(n) + 8$ matrices, and by the explicit definition of the Radon-Hurwitz function, if $n = (2a+1)2^{c+4d}$ then

$$\rho(16n) = \rho((2a+1)2^{c+4d+4}) = 2^c + 8(d+1) = \rho(n) + 8.$$

This gives an orthogonal system of the required order.

Definition 1.1.3 *Let $\sigma(n, n, n)$ be the maximum dimension of a linear space of $n \times n$ matrices, such that every non-zero matrix in the space is nonsingular.*

Radon-Hurwitz matrices give the lower bound $\rho(n)$ for $\sigma(n, n, n)$. That there can be no spaces of larger dimension is a consequence of Adams' solution of the vector fields on spheres problem [A1]. This proof relies on a considerable amount of homotopy and K-theory.

Remark In [R3] Rees shows that only the K-theoretic part of Adams' proof is required to obtain the upper bound $\sigma(n, n, n) \leq \rho(n)$. The simplification arises because we are concerned with *linear* maps, which have more structure than the continuous maps that form vector fields. A discussion of this method is deferred to chapter 2.

1.1.2 Vector fields on spheres

Let S^{n-1} be the unit sphere in R^n.

Definition 1.1.4 *A vector field on S^{n-1} is a continuous function v which assigns to each $x \in S^{n-1}$ a vector $v(x)$ tangent to S^{n-1} at x; r such fields $v_1, \ldots, v_r$, are said to be linearly independent if the vectors $v_1(x), \ldots, v_r(x)$ are linearly independent for all $x \in S^{n-1}$.*

The 'vector fields on spheres' problem is to determine the maximum number of linearly independent vector fields on S^{n-1}.

The existence of $r-1$ linearly independent vector fields on S^{n-1} is closely related to the existence of an orthogonal multiplication $f: R^r \times R^n \longrightarrow R^n$. Such a multiplication is said to be normalized providing $f(e_r, x) = x$ for each $x \in R^n$, where $e_r = (0, \ldots, 0, 1)$. In fact, given an orthogonal multiplication one can always construct a normalized multiplication (see [H2]). The next theorem shows how an orthogonal multiplication can be used to construct vector fields on a sphere.

Theorem 1.1.5 *If there exists an orthogonal multiplication $f : R^r \times R^n \longrightarrow R^n$ then there exist $r-1$ orthonormal vector fields on S^{n-1}.*

Proof By the above, f may be assumed to be normalized. For $x \in S^{n-1}$, the vectors $f(e_1, x), \ldots, f(e_{r-1}, x), x$ are orthonormal. The functions $v_i(x) = f(e_i, x)$ $(1 \leq i \leq r-1)$ are orthonormal vector fields on S^{n-1}.

Combining this with the Radon-Hurwitz Theorem, we conclude that there exist $\rho(n)-1$ linearly independent vector fields on S^{n-1}.

1.1.2.1 K-theory

K-theory was introduced by Grothendieck in the context of algebraic geometry. In the early 1960s, Atiyah and Hirzebruch [AH] constructed a topological analogue based on (suitably defined) equivalence classes of vector bundles over compact spaces. Standard references are [A5], [H2] and [K]. We recall the following basic definitions.

Definition 1.1.6 *The set of isomorphism classes of vector bundles over a compact space X form a monoid, denoted $Vect(X)$ under the relation $[E]+[F]=[E \oplus F]$. The symbol $\oplus$ represents the Whitney sum of bundles; the identity element is just the trivial 0-dimensional bundle.*

Definition 1.1.7 *$K(X)$, the 'K-theory of X', is the group completion of $Vect(X)$.*

The group completion of a monoid, being the solution of a universal problem, is unique (if it exists) up to isomorphism. Hence for a given compact space X, $K(X)$ will be uniquely defined up to isomorphism.

$Vect(X)$ can be given the structure of a semi-ring under the tensor product operation $[E] \times [F] = [E \otimes F]$; this factors through K-theory, to give $K(X)$ a ring structure.

The functor K is contravariant on the category of compact spaces. The projection of X onto a point P induces a homomorphism $\alpha : K(P) \longrightarrow K(X)$. Now $K(P) \cong Z$ (every bundle over a point is trivial, and so up to isomorphism there is only one bundle in every dimension). The cokernel of α is called the *reduced K-theory of* X and is denoted $\widetilde{K}(X)$. For X non-empty, there is a short exact sequence

$$0 \longrightarrow Z \longrightarrow K(X) \longrightarrow \widetilde{K}(X) \longrightarrow 0,$$

and the choice of a basepoint in X defines a corresponding splitting $K(X) \cong Z \oplus \widetilde{K}(X)$.

Returning to the vector fields problem, we state the main result.

Theorem 1.1.8 (Adams) *There do not exist $\rho(n)$ independent vector fields on S^{n-1}.*

Remark Writing n in the form $(2a+1)2^b$, the result for $b \leq 3$ is due to Steenrod and Whitehead [SW], and for $b \leq 10$ it is due to Toda [T1].

Corollary 1.1.9 *The linear spaces constructed by Radon and Hurwitz are of the largest possible dimension under the assumption that every non-zero matrix in the space is nonsingular. Hence $\sigma(n,n,n) = \rho(n)$.*

1.2 Complex, quaternionic and Hermitian spaces

We present a brief account of the results and methods used by Adams, Lax and Phillips [ALP] to study related problems on nonsingular matrices. (Note that a correction to the main result appears in [ALP2].)

Let Λ be either the real numbers R, the complex numbers C, or the skew-field of quaternions H. Denote by $\Lambda(n)$ the largest number of $n \times n$ matrices with entries in Λ, all of whose non-trivial real linear combinations are non-zero. Equivalently, $\Lambda(n)$ is the maximum dimension of a vector space of $n \times n$ matrices, such that every non-zero matrix in the space is nonsingular. Denote by $\Lambda_S(n)$ the largest number of Hermitian $n \times n$ matrices over Λ with this property. (For $\Lambda = R$, Hermitian just means symmetric.)

As explained in the previous section, Adams showed that $R(n) = \rho(n)$; $R_S(n)$, $C(n)$, $C_S(n)$, $H(n)$ and $H_S(n)$ can be determined by relating them to $R(n)$ through a series of inequalities. The results for $\Lambda = C$ are stated in terms of b, where $n = (2a+1)2^b$.

Theorem 1.2.1 (Adams, Lax and Phillips)

$$\begin{aligned} R(n) &= \rho(n) & \qquad R_S(n) &= \rho(n/2) + 1 \\ C(n) &= 2b + 2 & C_S(n) &= 2b + 1 \\ H(n) &= \rho(n/2) + 4 & H_S(n) &= \rho(n/4) + 5 \end{aligned}$$

Summary of Proof

Every real [symmetric] matrix may be regarded as a complex [Hermitian] matrix, which may in turn be regarded as a quaternionic [Hermitian] matrix. Hence

$$R(n) \leq C(n) \leq H(n) \qquad \text{and} \qquad R_S(n) \leq C_S(n) \leq H_S(n).$$

Also, by forgetting the complex [quaternionic] structure, an $n \times n$ matrix with entries in C $[H]$ may be thought of as a linear transformation of the underlying real [complex] vector space of dimension $2n$. Thus

$$C(n) \leq R(2n) \qquad \text{and} \qquad H(n) \leq C(2n).$$

Lemma 1.2.2

(i) $\Lambda(n) + 1 \leq \Lambda_S(2n)$.
(ii) $C_S(n) + 1 \leq C(n)$.
(iii) $H_S(n) + 3 \leq R(4n)$.
(iv) $R_S(n) + 3 \leq H(n)$.

Let V be a k-dimensional space of nonsingular $n \times n$ matrices over Λ and consider the R-linear map B defined by

$$B : \Lambda^n \oplus \Lambda^n \longrightarrow \Lambda^n \oplus \Lambda^n; \qquad (x, y) \mapsto (Ay + \lambda x, A^* x - \lambda y), \qquad (A \in V,\ \lambda \in R).$$

Here A^* is A if $\Lambda = R$, or represents the usual complex or quaternionic conjugation if $\Lambda = C$ or H. The matrix of the transformation B is given by

$$B = \begin{bmatrix} \lambda I_n & A \\ A^* & -\lambda I_n \end{bmatrix}.$$

The space so formed has dimension $k+1$ and consists entirely of Hermitian matrices. Suppose some B is singular, so there exists $(x, y) \in \Lambda^n \oplus \Lambda^n$, not both zero, such that $Ay + \lambda x = 0$ and $A^* x - \lambda y = 0$. Multiplying the first equation by x^* and the second by y^* gives, after subtraction , $\lambda(xx^* + yy^*) = 0$. Hence $\lambda = 0$ and so A must be singular and is therefore the zero matrix.

The inequality $C_S(n) \leq C(n) + 1$ is derived as follows. Let V be a k-dimensional space of $n \times n$ nonsingular Hermitian matrices, and λ be a purely imaginary complex number. Consider the $k+1$ dimensional space

$$A + \lambda I_n$$

If $A + \lambda I_n$ is singular then there exists a non-zero x such that $Ax = -\lambda x$. Then $-\lambda x^* x = x^* A x = (-\lambda x)^* x = \lambda x^* x$. Hence $\lambda = 0$ and so A must be singular, i.e. $A = 0$.

Parts (iii) and (iv) are proved in a similar manner. The results $C(n) = 2b + 2$ and $C_S(n) = 2b+1$ are obtained by combining parts (i) and (ii) with an induction argument.

To prove the remaining parts of the theorem, observe first that by using the explicit definition of the Radon-Hurwitz function, one can show that $\rho(n/2) + 1 = \rho(8n) - 7$. Using the lemma, we get $R(8n) - 7 = \rho(8n) - 7 = \rho(n/2) + 1 = R(n/2) + 1 \leq R_S(n)$. That is,

$$R_S(n) \geq R(8n) - 7.$$

Combining the inequalities in the lemma gives

$$\begin{aligned} R_S(n) &\leq H(n) - 3 && \text{(part (iv))} \\ &\leq H_S(2n) - 4 && \text{(part(i))} \\ &\leq R(8n) - 7 && \text{(part (iii))} \end{aligned}$$

which establishes the reverse inequality. Hence $R_S(n) = \rho(n/2)+1$, $H(n) = R_S(n)+3 = \rho(n/2) + 4$ and $H_S(2n) = H(n) + 1 = \rho(n/2) + 5$ and so $H_S(n) = \rho(n/4) + 5$.

Chapter 2

Linear Spaces of Fixed Rank

The earliest results on the dimensions (and possible forms) of fixed-rank linear subspaces of singular matrices are due to Westwick [W1]. He approached the problem from the point of view of algebraic geometry: the set of matrices whose rank is bounded above by a fixed number forms a variety, whose dimension is calculated by a well-known formula; the dimension of a fixed-rank subspace is clearly bounded above by the same number.

The loss of information involved in moving from fixed-rank spaces to bounded-rank spaces is, of course, significant. In [S2], Sylvester described a technique using vector bundles for obtaining upper bounds on spaces of fixed-rank complex matrices. The results obtained were a substantial improvement on the bounds implied by considering varieties. In particular, he was able to settle some outstanding questions from [W1]. Further results have since been obtained by Beasley [B], Atkinson [A7] and Westwick [AW],[W2],[W3],[W4].

More recently, a similar method has been used by Meshulam [M1], Rees [R2], and by Lam and Yiu [LY2] to obtain upper bounds on the dimensions of certain linear spaces of real matrices. An alternative proof (using only the K-theoretic part of Adams' argument [A1],[ALP]) of the fact that $\rho(n)$ is an upper bound for the dimension of a linear space of nonsingular $n \times n$ matrices is described by Rees.

2.1 Varieties of matrices

Let U and V be finite dimensional vector spaces over an algebraically closed field F and denote by $L(U, V)$ the space of linear transformations from U to V. If U and V have dimensions n and m respectively then we can interpret $L(U, V)$ as the space of $m \times n$ matrices with entries in F, denoted $M_{mn}(F)$.

Theorem 2.1.1 (Westwick) *Let m, n, k be integers with $k \leq min(m, n)$. The matrices of rank $\leq k$ form an irreducible variety of dimension $mn - (m-k)(n-k)$ in $M_{mn}(F)$.*

Remark This is, of course, a familiar theorem in algebraic geometry. Westwick's proof, however, makes no assumptions about the characteristic of F, whereas the proofs in [R5] and [M2] apply only to fields of characteristic zero.

Example The variety of matrices of rank $\leq n-2$ in $M_{nn}(F)$ has dimension n^2-4 and so 4 is an upper bound for the dimension of linear spaces of fixed-rank $n-1$. Westwick [W1] constructs examples to show that there exists a 3 dimensional space when $n \geq 3$ is odd, and a 2 dimensional space for n even.

2.2 A topological approach

This exposition follows that of [R2], where it is described for linear spaces of real matrices. The construction in [S2] for complex linear spaces is similar.

Let F be either the field R of real numbers or the field C of complex numbers, and define $F^* := F\backslash\{0\}$. Let $V \subset M_{mn}(F)$ be a linear subspace of dimension $d+1$ with the property that every non-zero matrix A in V has fixed rank k. Consider the map

$$\Phi : V\backslash\{0\} \times F^n \longrightarrow V\backslash\{0\} \times F^m; \qquad (A, \underline{x}) \mapsto (A, A\underline{x}).$$

Define actions T_1, T_2 of F^* on $V\backslash\{0\} \times F^n$ and $V\backslash\{0\} \times F^m$ by

$$\begin{aligned} T_1(\alpha, (A, \underline{x})) &= (\alpha A, \underline{x}) \qquad (\alpha \in F^*,\ A \in V\backslash\{0\},\ \underline{x} \in F^n), \\ T_2(\alpha, (A, \underline{x})) &= (\alpha A, \alpha\underline{x}) \qquad (\alpha \in F^*,\ A \in V\backslash\{0\},\ \underline{x} \in F^m). \end{aligned}$$

Then T_1, T_2 and Φ satisfy

$$\Phi T_1 = T_2 \Phi.$$

Notation Denote by λ_V the tautological line bundle over the projective space $P(V)$. If $S(V)$ denotes the set of vectors of unit length in V, then the total space of λ_V is

$$(S(V) \times F)/(\underline{x}, \alpha) \sim (z\underline{x}, z\alpha), \qquad (z \in F,\ |z| = 1).$$

(For background material on vector bundles see [ES], [MS] or [A5].)

V has dimension $d+1$, so we will just write FP^d for $P(V)$ and λ for λ_V. Denote the trivial n-dimensional bundle over FP^d by $n\epsilon = \epsilon \oplus \epsilon \oplus \ldots \oplus \epsilon$. Since Φ is equivariant with respect to T_1, T_2, we obtain a homomorphism of vector bundles

$$\Phi : \ n\epsilon \longrightarrow m\lambda.$$

Since every non-zero matrix in V has fixed rank k then we can define bundles

$$F^k := \ \mathrm{Im}(\Phi), \qquad G^{n-k} := \ \mathrm{Ker}(\Phi), \qquad H^{m-k} := \ \mathrm{Coker}(\Phi)$$

of dimensions $k, n-k$ and $m-k$ respectively.

The homomorphism Φ gives a short exact sequence of vector bundles over FP^d

$$0 \longrightarrow G^{n-k} \longrightarrow n\epsilon \xrightarrow{\Phi} m\lambda \longrightarrow H^{m-k} \longrightarrow 0.$$

Every such exact sequence splits (see [A5]), thus giving an isomorphism of bundles

$$G^{n-k} \oplus m\lambda \cong H^{m-k} \oplus n\epsilon.$$

We can rewrite the exact sequence as two short exact sequences to obtain

$$F^k \oplus G^{n-k} \cong n\epsilon \qquad \text{and} \qquad F^k \oplus H^{m-k} \cong m\lambda.$$

Philosophy We have shown that if V is a $d+1$-dimensional linear space of fixed-rank matrices with entries in R or C then we obtain a homomorphism of vector bundles over RP^d or CP^d and a short exact sequence. The idea is to investigate whether such an exact sequence can exist for a given d. For the complex case, calculations with Chern classes are used; Stiefel-Whitney classes and K-theory are the corresponding tools for real matrices. If a contradiction is obtained then we have the upper bound d for the maximum dimension.

Definition 2.2.1 *For positive integers $m, n \geq k$, let $\sigma_C(m,n,k)$ be the largest dimension of linear spaces of $m \times n$ matrices with entries in C, all of whose non-zero matrices have fixed rank k; for real matrices we omit the subscript and write $\sigma(m,n,k)$.*

2.3 Complex spaces

For a bundle ζ over CP^d write the total Chern class $c(\zeta)$ as

$$c(\zeta) = 1 + c_1 x + c_2 x^2 + \ldots + c_d x^d$$

where $c(\zeta) \in H^*(CP^d; Z)$, which is the truncated polynomial ring $Z[x]$ with $x^{d+1} = 0$. For bundles ζ_1, ζ_2 over the same base space then the total Chern class satisfies

$$c(\zeta_1 \oplus \zeta_2) = c(\zeta_1)c(\zeta_2).$$

For the tautological line bundle λ we have $c(\lambda) = 1 + x$ and so $c(m\lambda) = (1+x)^m$ mod x^{d+1}; for a trivial bundle ϵ then $c(\epsilon) = 1$. Apply the product theorem to the bundle isomorphisms given by the short exact sequence splittings to get

$$\begin{aligned} c(F^k)c(G^{n-k}) &= 1 && \text{mod } x^{d+1}, \\ c(F^k)c(H^{m-k}) &= (1+x)^m && \text{mod } x^{d+1}, \\ c(G^{n-k})(1+x)^m &= c(H^{m-k}) && \text{mod } x^{d+1}. \end{aligned}$$

This method was used by Sylvester to prove that the spaces of $n \times n$ complex matrices of rank $n-1$ constructed in [W1] are of maximal dimension.

Theorem 2.3.1 (Sylvester)

$$\sigma_C(n, n, n-1) = \begin{cases} 2 & \textit{for } n \textit{ even.} \\ 3 & \textit{for } n \textit{ odd, } n > 1. \end{cases}$$

Proof The cases n even and n odd are considered separately.

Case $n = 2p$. Assume that $\sigma_C(2p, 2p, 2p-1) \geq 3$. Then there exist complex line bundles G^1 and H^1 over CP^2 satisfying

$$c(G^1)(1+x)^{2p} = c(H^1) \qquad \text{mod } x^3.$$

G^1, H^1 have dimension 1, so $c_2(G^1) = c_2(H^1) = 0$. We proceed by expressing the class $c_1(G^1) \in H^2(CP^2; Z)$ in terms of p. This is achieved by writing $c(G^1) = 1 + gx$ for some integer g and comparing coefficients. To obtain a contradiction we will show that g cannot be an integer.

The coefficients of x^2 in our relation give

$$2pg + \binom{2p}{2} = 0.$$

This gives $g = \frac{1-2p}{2}$, which is the required contradiction.

Case $n = 2p+1,\ p > 0$. Assume that $\sigma_C(2p+1, 2p+1, 2p) \geq 4$. This gives the following relation in $H^*(CP^3; Z)$

$$c(G^1)(1+x)^{2p+1} = c(H^1) \qquad \text{mod } x^4.$$

As for the previous case, we write $1 + gx$ for $c(G^1)$ and compare coefficients of x^2 to obtain the equation $g = -p$. The coefficient of x^3 must also be zero. That is,

$$\binom{2p+1}{2}g + \binom{2p+1}{3} = 0.$$

Substituting for g gives (after some manipulation) the equation $p(p+1) = 0$, which has no positive integer solutions. This completes the proof.

Notice that $M_{mn}(C) \cong M_{nm}(C)$, and so $\sigma_C(m,n,k) = \sigma_C(n,m,k)$.

Proposition 2.3.2 (Sylvester) *For* $m \geq k$, $\sigma_C(m,k,k) = m - k + 1$.

Proof The following band matrix has maximal rank and is of the required dimension.

$$\begin{bmatrix} a_1 & a_2 & \dots & a_{m-k+1} & 0 & 0 & \dots & 0 \\ 0 & a_1 & a_2 & \dots & a_{m-k+1} & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \ddots & & \vdots \\ 0 & 0 \dots & 0 & a_1 & a_2 & \dots & \dots & a_{m-k+1} \end{bmatrix}.$$

Suppose $\sigma_C(k,m,k)$ $(= \sigma_C(m,k,k)) \geq m - k + 2$. Then we must have $c(F^k) = 1$ and $c(H^{m-k}) = (1+x)^m$ in $H^*(CP^{m-k+1}; Z)$.

All the binomial coefficients in the expansion of $(1+x)^m$ are non-zero, and in particular, the coefficient of x^{m-k+1} is non-zero. But then $c_{m-k+1}(H^{m-k}) \neq 0$, which contradicts the fact that H^{m-k} is a bundle of dimension $m-k$.

Remark Sylvester also uses a similar method to investigate some linear subspaces of $L(C^n, C^m)$ of bounded rank. The interested reader is referred to [S2].

The above technique was used by Westwick in [W2] and [W3] to prove the following general result.

Theorem 2.3.3 (Westwick) *For* $m \geq n \geq k \geq 2$,

(i) $m - k + 1 \leq \sigma_C(m, n, k) \leq (m - k) + (n - k) + 1$.
(ii) $\sigma_C(m, n, k) = m - k + 1$ *whenever* $m - k + 1$ *does not divide* $(n-1)!/(k-1)!$.
(iii) $\sigma_C(rk + k - 1, rk + 1, rk) = k + 1$ *for each positive integer* r.

Summary of proof This theorem solves the problem for complex matrices for a very large number of cases. The previous construction (adding rows and columns of zeros as necessary) provides the lower bound for (i). The upper bound is obtained by considering the general situation for a space V of $m \times n$ complex matrices of fixed rank k. Let $d+1$ be the dimension of V, so there exists a relation in $H^*(CP^d; Z)$

$$c(G^{n-k})(1+x)^m = c(H^{m-k}) \qquad \bmod x^{d+1}.$$

Write $c(G^{n-k}) = 1 + g_1 x + \ldots + g_{n-k}x^{n-k}$ and let

$$(1 + g_1 x + \ldots + g_{n-k}x^{n-k})(1+x)^m = 1 + p_1 x + p_2 x^2 + \ldots$$

Then the coefficients p_i can be written

$$p_i = \tbinom{m}{i} + \tbinom{m}{i-1} g_1 + \ldots + \tbinom{m}{i-n+k} g_{n-k}.$$

If $d + 1 \geq (m-k) + (n-k) + 2$ then we must have

$$p_{m-k+1} = p_{m-k+2} = \ldots = p_{m+n-2k+1} = 0.$$

One can check (see [MJ]) that the matrix of coefficients associated with this system of linear equations is nonsingular. But $c(G^{n-k})$ has constant term 1, which gives a contradiction and proves part (i). Part (ii) implies that the upper bound in (i) can usually be improved upon. It is equivalent to the statement that the equation

$$p_{m-k+1} = 0$$

has no integer solutions whenever $(n-1)!/(k-1)!$ is not a multiple of $m - k + 1$. Part (iii) is a special case where this condition is not satisfied and is proved in [W3]; the case $r = 1$ is due to Beasley [B]. Other examples are given in [A7],[AW], and [W4].

2.4 Real spaces

In this section we summarize the methods and results of Rees [R2],[R3] (see also [M1],[LY2]) on fixed-rank linear spaces of real matrices. Recall that the largest dimension of a space of $m \times n$ real matrices of rank k is denoted by $\sigma(m, n, k)$. The first proposition is just the real version of observations made in the last section.

Proposition 2.4.1 *For* $m \geq n \geq k$,
(i) $\sigma(m, n, k) = \sigma(n, m, k)$.
(ii) $\sigma(m + 1, n, k) \geq \sigma(m, n, k)$.
(iii) $\sigma(m, k, k) \geq m - k + 1$.

Proof (i) follows from the isomorphism of vector spaces $M_{mn}(R) \cong M_{nm}(R)$; for (ii), add a row of zeroes to every $m \times n$ matrix to obtain a space of $(m+1) \times n$ matrices of the same rank; (iii) follows from the band matrix of proposition 2.3.2.

Recall that if $V \subset M_{mn}(R)$ is a $d+1$-dimensional linear space with the property that $A \in V \backslash \{0\} \Rightarrow \text{rank}(A) = k$, then there exists a homomorphism $\Phi : n\epsilon \longrightarrow m\lambda$ of real vector bundles over RP^d and an exact sequence of the form

$$0 \longrightarrow G^{n-k} \longrightarrow n\epsilon \xrightarrow{\Phi} m\lambda \longrightarrow H^{m-k} \longrightarrow 0,$$

where G^{n-k} is the $n-k$-dimensional bundle $\text{Ker}(\Phi)$ and H^{m-k} is the $m-k$-dimensional bundle $\text{Coker}(\Phi)$. The splitting gives an isomorphism

$$G^{n-k} \oplus m\lambda \cong H^{m-k} \oplus n\epsilon,$$

and if F^k is the k-dimensional bundle $\text{Im}(\Phi)$, then

$$F^k \oplus G^{n-k} \cong n\epsilon \qquad \text{and} \qquad F^k \oplus H^{m-k} \cong m\lambda.$$

Chern classes were used to analyze the complex bundles of the last section. The analogous tool for real bundles is the theory of Stiefel-Whitney classes (see [MS]). The mod 2 cohomology of RP^d is given by

$$H^i(RP^d; Z_2) = \begin{cases} Z_2 & 1 \leq i \leq d. \\ 0 & i > d. \end{cases}$$

As an algebra $H^*(RP^d; Z_2)$ is generated by the non-zero element $x \in H^1(RP^d; Z_2)$, subject to the relation $x^{d+1} = 0$.

Write the total Stiefel-Whitney class of a r-dimensional bundle ζ as

$$w(\zeta) = 1 + w_1 x + w_2 x^2 + \ldots + w_r x^r.$$

As for Chern classes, there is a product theorem: for bundles ζ_1, ζ_2 over the same base space then

$$w(\zeta_1 \oplus \zeta_2) = w(\zeta_1) w(\zeta_2).$$

Theorem 2.4.2 (Meshulam and Rees) *$\sigma(m, n, k) \leq max(m, n)$.*

Proof Assume $m \geq n$ and $\sigma(m, n, k) \geq m + 1$, so there exist isomorphisms over RP^m

$$F^k \oplus G^{n-k} \cong n\epsilon \qquad \text{and} \qquad F^k \oplus H^{m-k} \cong m\lambda.$$

Now $w_m(m\lambda) \neq 0$, so (by the product theorem) both $w_{m-k}(H^{m-k})$ and $w_k(F^k)$ are non-zero. Let $r \leq n - k$ be the highest non-zero Stiefel-Whitney class of G^{n-k}. Then $r + k \leq (n - k) + k = n \leq m$ and

$$w_{r+k}(F^k \oplus G^{n-k}) = w_k(F^k) w_r(G^{n-k}) \neq 0.$$

This gives a contradiction: the bundle $F^k \oplus G^{n-k}$ is isomorphic to a trivial bundle, so all its Stiefel-Whitney classes vanish.

Theorem 2.4.3 (Rees)
(i) $\sigma(m, n, 1) = max(m, n)$.
(ii) $\sigma(n, n, 2) = \begin{cases} n & \textit{for } n \textit{ even.} \\ n - 1 & \textit{for } n \textit{ odd}, \; n > 3; \; \sigma(3, 3, 2) = 3. \end{cases}$

Proof (i) follows from theorem 2.4.2 and proposition 2.4.1(iii). For (ii), write $n = 2p$ for n even and $n = 2p + 1$ for n odd. Every non-zero matrix in the following space has rank 2

$$\begin{bmatrix} a_1 & -b_1 & a_2 & -b_2 & \ldots & a_p & -b_p \\ b_1 & a_1 & b_2 & a_2 & \ldots & b_p & a_p \end{bmatrix}.$$

Hence $\sigma(2p, 2, 2) \geq 2p$ and by proposition 2.4.1(ii), $\sigma(2p, 2p, 2) \geq 2p$; by theorem 2.4.2, we must have equality. Also, $\sigma(2p + 1, 2p + 1, 2) \geq \sigma(2p, 2p, 2) = 2p$; a Stiefel-Whitney class argument shows that for $p > 1$ no space of dimension $2p + 1$ exists, and the example on page 1 shows that $\sigma(3, 3, 2) \geq 3$ (and by the above must equal 3).

2.4.1 More K-theory

For spaces where the fixed rank is very large, K-theory often gives stronger information than Stiefel-Whitney classes. In [A1], Adams calculated the structure of $\widetilde{KO}(RP^d)$. To state the result we need the following definition.

Definition 2.4.4 *$\phi(d)$ counts the number of integers congruent to $0, 1, 2$ or 4 mod 8 in the interval $[1, d]$. Explicitly, $\phi(1) = 1$, $\phi(2) = \phi(3) = 2$, $\phi(4) = \ldots = \phi(7) = 3$; for $d \geq 1$, $\phi(8d) = 4d$, $\phi(8d+1) = 4d+1$, $\phi(8d+2) = \phi(8d+3) = 4d+2$, $\phi(8d+4) = \ldots = \phi(8d+7) = 4d+3$.*

Fact As a group, $\widetilde{KO}(RP^d)$ is isomorphic to the cyclic group of order $2^{\phi(d)}$. The group is generated by the image of the tautological line bundle λ over RP^d, which we denote by x. As a ring, $\widetilde{KO}(RP^d)$ is isomorphic to $Z[x]$ modulo the relations $x^2 = -2x$ and $x^{\phi(d)+1} = 0$. (The relation $x^2 = -2x$ is a consequence of the fact that $\lambda \otimes \lambda$ is trivial over RP^d.) Consult Adams [A1] for further details.

Definition 2.4.5 *If $\rho(n)$ denotes the Radon-Hurwitz function then*

$$\rho(n, n-k) = max(\rho(n-k), \ldots, \rho(n), \ldots, \rho(n+k)).$$

Theorem 2.4.6 (Rees, Lam and Yiu)
(i) $\sigma(n, n, n) = \rho(n)$.
(ii) $\sigma(n, n, n-1) = \rho(n, n-1)$, $n > 3$, $n \neq 7$; $\sigma(3, 3, 2) = 3$, $\sigma(7, 7, 6) = 7$.
(iii) $\sigma(n, n, n-2) = \rho(n, n-2)$, $n > 3$, $n \neq 6, 7$; $\sigma(6, 6, 4) = \sigma(7, 7, 5) = 6$.

Remark The original solution of part (i) (see [ALP]) is described in chapter 1: recall that for each n we can construct a $\rho(n)$-dimensional space of $n \times n$ Radon-Hurwitz matrices; the fact that there can be no space of higher dimension follows from Adams' solution of the vector fields on spheres problem. The proof given here (due to Rees [R2], and more explicitly in [R3]) of the upper bound $\sigma(n, n, n) \leq \rho(n)$ uses only the K-theoretic part of Adams' proof.

Summary of proof

First we show that $\rho(n, n-k)$ is an upper bound. The cases $k = 0, 1, 2$ are considered separately. The following simple lemma is easily verified.

Lemma 2.4.7 *Let $v_2(n)$ be the exponent of the highest power of 2 dividing n and let $\rho(n)$ be the Radon-Hurwitz function. Then $v_2(n) = \phi(\rho(n) - 1) = \phi(\rho(n)) - 1$.*

Case $k = 0$. Let V be a space of $n \times n$ nonsingular matrices of dimension $d+1$. The homomorphism $\Phi : n\epsilon \longrightarrow n\lambda$ of bundles over RP^d is in this case an isomorphism and so $n\lambda$ is trivial over RP^d. In $\widetilde{KO}(RP^d)$ we have $nx = 0$ and hence

$$n \equiv 0 \bmod 2^{\phi(d)}.$$

Now $n \equiv 0 \bmod 2^{\phi(d)} \Rightarrow v_2(n) \geq \phi(d)$. By the lemma, $\phi(\rho(n)) - 1 \geq \phi(d)$, or $\phi(d)+1 \leq \phi(\rho(n))$. But ϕ is monotonic increasing, so $d+1 \leq \rho(n)$ and $\sigma(n,n,n) \leq \rho(n)$.

Case $k = 1$. The existence of a $d+1$-dimensional space of $n \times n$ real matrices, all of whose non-zero entries have rank $n-1$ gives an exact sequence

$$0 \longrightarrow G^1 \longrightarrow n\epsilon \xrightarrow{\Phi} n\lambda \longrightarrow H^1 \longrightarrow 0$$

on RP^d, and an isomorphism

$$G^1 \oplus n\lambda \cong H^1 \oplus n\epsilon.$$

For any $d \geq 1$, every real line bundle over RP^d must be isomorphic either to the trivial bundle ϵ or to the tautological line bundle λ. (This is because $BO(1)$ is an Eilenberg-MacLane space of type $(Z_2, 1)$, and so $H^1(RP^d; Z_2) \cong Z_2$ classifies real line bundles.) Hence there are four choices (up to isomorphism) for the ordered pair (G^1, H^1).

If $G^1 \cong \epsilon$ and $H^1 \cong \lambda$ then $(n-1)x = 0$ in $\widetilde{KO}(RP^d)$ and so $d+1 \leq \rho(n-1)$.

If $G^1 \cong \epsilon$ and $H^1 \cong \epsilon$ or $G^1 \cong \lambda$ and $H^1 \cong \lambda$ then $nx = 0$ and $d+1 \leq \rho(n)$.

If $G^1 \cong \lambda$ and $H^1 \cong \epsilon$ then $(n+1)x = 0$ and $d+1 \leq \rho(n+1)$.

Hence for all n we have $\sigma(n,n,n-1) \leq \rho(n,n-1)$. The special case $\sigma(3,3,2)$ is covered by theorem 2.4.3; for $n = 7$, we have $\rho(7,6) = 8$, but theorem 2.4.2 gives $\sigma(7,7,6) \leq 7$.

Case $k = 2$. The corresponding isomorphism is $G^2 \oplus n\lambda \cong H^2 \oplus n\epsilon$.

Fact For $d \geq 3$, every $O(2)$ bundle over RP^d is isomorphic to one of $2\epsilon, \epsilon \oplus \lambda$ or 2λ. (See Adams [A2].)

Hence for $n > 3$ the same approach as for $k = 1$ yields $\sigma(n,n,n-2) \leq \rho(n,n-2)$. For $n = 6$, $\sigma(6,6,4) \leq 6$ by theorem 2.4.2; and for $n = 7$, one can show that a 7-dimensional space cannot exist (see [R2]).

Constructing large-rank spaces

Let V be a real vector space equipped with a negative definite inner product. We can associate to V its Clifford algebra, denoted $C(V)$ (see [ABS],[H2],[K]).

Let $k-1$ be the dimension of V. Following [K], R^n is a $C(V)$ module if and only if there exists an orthogonal multiplication $R^k \times R^n \longrightarrow R^n$, that is if and only if there exist $k-1$ orthonormal tangent vector fields on S^{n-1}. Hence by theorem 1.1.8, for R^n to be a $C(V)$ module then we must have $\dim(V) \leq \rho(n) - 1$. Define V_1 to be the span of $1 \in C(V)$ and $V \subset C(V)$. Then $\dim(V_1) = \dim(V) + 1 \leq \rho(n)$. We will always take V to be of maximal dimension, i.e. $\dim(V_1) = \rho(n)$.

If the n-dimensional vector space W is a $C(V)$ module, and $z \in V_1$, then denote by L_z the linear transformation of W induced by multiplication by z. If z^* is the conjugate of z then for a suitable inner product on W, L_z and L_{z^*} will be adjoint transformations, and $L_z L_{z^*} = L_{z^*} L_z = \| z \|^2$.

Theorem 2.4.8 (Rees, Lam and Yiu) *Suppose* $\dim(W) = n$ *and* $\dim(V_1) = \rho(n)$. *If* $X, Y \subset W$ *and* $U \subset V_1$ *are subspaces of dimensions* s, t *and* l *respectively, and such that* $Y \perp UX$, *then* $\sigma(n-s, n-t, n-s-t) \geq l$.

Summary of proof

Let $\pi : W \to Y^\perp$ denote orthogonal projection. Consider the composition

$$P_z : X^\perp \hookrightarrow W \xrightarrow{L_z} W \xrightarrow{\pi} Y^\perp \qquad (z \in V_1).$$

Each P_z is a linear transformation from R^{n-s} to R^{n-t}. If P_z has rank $n-s-t$ for every non-zero $z \in V_1$ then $\sigma(n-t, n-s, n-s-t)$ $(= \sigma(n-s, n-t, n-s-t))$ $\geq l$.

Lemma 2.4.9
(i) $Ker(\pi L_z) = L_{z^*} Y$.
(ii) If $Y \perp L_z(X)$ *then* $Ker(\pi L_z) \subset X^\perp$.

Proof of lemma For $y \in Y$, $L_z(L_{z^*} y) = \| z \|^2 y$, which maps to 0 under π; if $x \in \mathrm{Ker}(\pi L_z)$, then $L_z x \in Y$ and so $L_{z^*}(L_z x) \in L_{z^*} Y$, that is $\| z \|^2 x \in L_{z^*} Y$ and so $x \in L_{z^*} Y$, which proves (i). For (ii), suppose that $Y \perp L_z(X)$ and let $x \in X$. Since L_z and L_{z^*} are adjoint transformations, $< L_{z^*} y, x > = < y, L_z x > = 0$. Hence $L_{z^*} Y \subset X^\perp$.

Now $\dim(\mathrm{Ker}(\pi L_z)) = \dim(L_{z^*} Y) = \dim(Y) = t$ since L_z is injective. Also $\mathrm{Ker}(\pi L_z) \subset X^\perp$, so by the dimension theorem, the image of P_z must have dimension equal to $\dim(X^\perp) - \dim(Y)$. In other words, for every non-zero $z \in V_1$, P_z has rank $n-s-t$.

Philosophy To give a strong lower bound using the above method, l must be large. That is, $U \subset V_1$ must be as large as possible. We also require $Y \perp UX$. So we look for subspaces $U \subset V_1$ such that U is large but UX is small. The combination $U = V_1$ and $Y = (UX)^{\perp}$ is successful if W is sufficiently large. For some low-dimensional examples, a more subtle choice of U must be made. This motivates the following definition.

Definition 2.4.10 *Let $\theta(n, s)$ be the minimum dimension for V_1X as X varies over all the s-dimensional subspaces of an n-dimensional real vector space W.*

Proposition 2.4.11

(i) $s \leq \theta(n, s) \leq n$ for all $s \leq n$.
(ii) $\theta(n, s) \leq \theta(n, s+1)$ for all $s < n$.
(iii) $\theta(n_1, s) \leq \theta(n_2, s)$ for $\rho(n_1) \leq \rho(n_2)$.
(iv) $\theta(n, 1) = \rho(n)$ for all n.
(v) $\theta(n, s) = n$ for $n = 1, 2, 4$ or 8 and $s \leq n$.
(vi) $\theta(2n, s) \leq \theta(n, s) + s(\rho(2n) - \rho(n))$ for all $s \leq n$.

Proof

Since $1 \in V_1$ then $X \subset V_1X$; also $V_1X \subset W$, so $s \leq \theta(n, s) \leq n$, which proves (i); Parts (ii) and (iii) are also immediate since θ is clearly weakly monotonic in s and $\rho(n)$.

If W is an n-dimensional vector space then $\dim(V_1) = \rho(n)$. If $X \subset W$ is a 1-dimensional subspace then clearly V_1X also has dimension $\rho(n)$, which proves (iv).

Part (v) is a consequence of the fact that multiplication by the real numbers on R, by the complex numbers on R^2, by the quaternions on R^4 and by the Cayley numbers on R^8 is nonsingular.

Suppose that $X \subset W$ is a s-dimensional subspace such that V_1X has dimension $\theta(n, s)$ (V_1 has dimension $\rho(n)$). Let V_1' be of dimension $\rho(2n)$ and thus act on R^{2n}. Then the number of extra dimensions of $V_1'X$ is at most $s(\rho(2n) - \rho(n))$, which proves (vi).

Further properties of θ are given in [R2], but will not be used in this thesis.

Returning to theorem 2.4.6, we now construct the spaces of $n \times n$ matrices of dimension $\rho(n, n-1)$. (The nonsingular matrices of dimension $\rho(n)$ are described in chapter 1.) There are 3 main cases.

Case $n \equiv 1$ mod 4. Here $\rho(n, n-1) = \rho(n-1)$. We can use proposition 2.4.1(ii): $\sigma(n, n, n-1) \geq \sigma(n-1, n-1, n-1) = \rho(n-1)$. To construct the space, begin with a space of $n-1 \times n-1$ matrices of dimension $\rho(n-1)$; simply add an extra row and column of zeroes to obtain a space of $n \times n$ matrices of rank $n-1$.

Case n even. Here $\rho(n, n-1) = \rho(n)$. Let $\dim(W) = n$, and take $U = V_1$, $s = 1$ and $t = 0$. Then since $Y = \{0\}$, $V_1 X \perp Y$ and so $\sigma(n-1, n, n-1) \geq \rho(n)$ and hence $\sigma(n, n, n-1) \geq \rho(n)$. To construct such a space, take a space V of nonsingular $n \times n$ matrices of dimension $\rho(n)$. Consider the map α, which replaces the first row of each $A \in V$ by a row of zeros and leaves the remaining $n-1$ rows unaltered. Then α is injective, since its kernel contains only the zero matrix. If $A_1, \ldots, A_{\rho(n)}$ is a basis for V, then $\alpha(A_1), \ldots, \alpha(A_{\rho(n)})$ form a basis for the new vector space. There can be no linear dependence amongst the $\alpha(A_i)$, for this would imply the existence of a rank 1 matrix in V. Hence the resulting vector space has the same dimension, $\rho(n)$, as V.

Remark A straightforward generalization of the above argument shows that a space of $n \times n$ nonsingular matrices can be used to construct, for each $k \leq n$, a space of $n \times n$ rank k matrices of the same dimension. Hence $\sigma(n, n, k) \geq \rho(n)$.

Case $n \equiv 3$ mod 4. Here $\rho(n, n-1) = \rho(n+1)$. Let $\dim(W) = n+1$, and $s = t = 1$. If $U = V_1$ then $\dim(V_1 X) = \rho(n+1)$. For $V_1 X$ to be orthogonal to Y we require $\theta(n+1, 1) + 1 \leq n+1$. That is, $\rho(n+1) \leq n$, which is true for all $n \equiv 3$ mod 4 except $n = 3$ and $n = 7$. Hence for $n > 7$ we have $\sigma(n, n, n-1) \geq \rho(n+1)$. For $n = 3$ we have already seen that $\sigma(3, 3, 2) = 3$. For $n = 7$ then $\dim(V_1) = \dim(W) = 8$; take $U \subset V_1$ to be of codimension 1 (so $\dim(UX) = 7$) and $Y = (UX)^{\perp}$ to get $\sigma(7, 7, 6) \geq 7$.

This completes the proof of part (ii) of theorem 2.4.6. A similar approach can be used to construct spaces of rank $n-2$ (see [R2] for details).

In theorem 2.4.6, the key to obtaining the upper bound $\sigma(n, n, n-1) \leq \rho(n, n-1)$ is the fact that every $O(1)$ bundle over RP^d is isomorphic either to the trivial bundle ϵ or to the tautological line bundle λ. Similarly, the fact that the only $O(2)$ bundles (up to isomorphism) over RP^d for $d \geq 3$ are $2\epsilon, \epsilon \oplus \lambda$, and 2λ, is used to obtain the bound $\sigma(n, n, n-2) \leq \rho(n, n-2)$. The situation for $O(n)$ bundles over RP^d is much more complicated for $n > 2$. One must consider the *geometric dimension* of a bundle, or equivalently, of its image in K-theory.

Definition 2.4.12 *Let ζ be an $O(n)$ bundle over X. The geometric dimension of ζ is the minimum fibre dimension of all bundles stably equivalent to ζ.*

In other words, ζ has geometric dimension r over X if r is the smallest positive integer for which there exists a bundle η of dimension r over X such that $\zeta \cong \eta \oplus (n-r)\epsilon$.

It is usual to consider the geometric dimension of elements of $\widetilde{KO}(X)$. For $X = RP^d$ the group is generated by the virtual bundle x, which is the image of the tautological line bundle λ over RP^d. The geometric dimension of the element mx is the minimum fibre dimension of all bundles over RP^d whose image in $\widetilde{KO}(X)$ is mx. This situation has been studied by Adams in [A2]. In chapter 3 we use these and other results on geometric dimension of Lam and Randall ([LR1],[LR2],[LR3]) to investigate further the dimensions of linear spaces where the common rank is large.

2.4.2 Related problems on bilinear maps

We have seen in chapter 1 that the existence of systems of orthogonal matrices is equivalent to the existence of certain normed bilinear maps. Suppose now that we impose a weaker restriction, namely that a bilinear map is merely nonsingular:

Definition 2.4.13 *A bilinear map*

$$\phi : R^a \times R^b \longrightarrow R^c$$

is said to be nonsingular if $\phi(x, y) = 0 \Rightarrow x = 0$ or $y = 0$.

Proposition 2.4.14 *Let a, b, c be positive integers with $c \geq max(a, b)$. There exists a nonsingular bilinear map $R^a \times R^b \longrightarrow R^c \iff \sigma(c, a, a) \geq b \iff \sigma(c, b, b) \geq a$.*

Proof. Let a, b and c be as above and suppose there is a nonsingular bilinear map $\phi : R^a \times R^b \longrightarrow R^c$. For each $x \in R^a \backslash \{0\}$ we can define a linear transformation

$$\phi_x : R^b \longrightarrow R^c; \quad y \mapsto \phi(x, y).$$

We can interpret each ϕ_x as a $c \times b$ matrix of rank b; these matrices form a vector space of dimension a. Hence $\sigma(c, b, b) \geq a$. Of course, we can do the same with the other factor and get $\sigma(c, a, a) \geq b$.

Now assume $\sigma(c, b, b) \geq a$. Then there exists a linear space $V \cong R^a$ of $c \times b$ matrices, all of whose non-zero entries have maximal rank b. Regard each non-zero $A \in V$ as a map $R^b \longrightarrow R^c$. Now define

$$\phi : \ V \times R^b \longrightarrow R^c; \quad (A, \mathbf{v}) \mapsto A\mathbf{v}.$$

It is easy to check that ϕ has the required properties.

Corollary 2.4.15 *$\sigma(n,k,k)$ is the largest positive integer r for which there exists a nonsingular bilinear map $R^r \times R^k \longrightarrow R^n$.*

In chapter 4 we make use of this connection to calculate the maximum dimensions of some spaces where the rank is relatively small.

Much of the literature on bilinear maps and related problems is due to K.Y. Lam and J. Adem (see [L1],[L2],[L3],[A3],[A4]). The non-existence of these maps is usually proved by considering homomorphisms of certain vector bundles. Examples of applications to immersion and embedding theorems for projective spaces can be found in [A6],[G1],[G2],[S1].

Chapter 3

Spaces of Large Rank

Recall that the function $\rho(n, n-k)$ is defined as the maximum of the $2k+1$ integers $\rho(n-k), \ldots, \rho(n), \ldots, \rho(n+k)$. In [R2] and [LY2] it is shown that if $n \geq 8$ and $k = 1$ or 2 then $\sigma(n, n, n-k) = \rho(n, n-k)$. In this chapter we investigate whether this relation continues to hold as k is increased.

Sections 1 and 2 respectively consider the cases $k = 3$ and 4. We show that the relation holds for most values of n. For $k = 3$, the only possible exceptions are when n is congruent to $64 \pm \epsilon$ mod 128, with $0 \leq \epsilon \leq 2$.

Unfortunately, the methods used here do not apply to these 5 exceptional cases. It seems clear, however, that they will all be proved together. A sufficient condition for these cases is formulated in terms of lower bounds for the geometric dimensions of certain elements of $\widetilde{KO}(RP^{12})$.

In section 3 we discuss the limitations of the conjecture that $\sigma(n, n, n-k) = \rho(n, n-k)$ for n sufficiently large. It is shown that for every k, we can find an n_k such that for all $n \geq n_k$, we have $\sigma(n, n, n-k) \geq \rho(n, n-k)$. For $5 \leq k \leq 7$ we exhibit a few further cases where equality occurs. For $k = 8$ our techniques yield no information, and for $k = 9$ there are infinitely many cases where we can construct spaces of dimension exceeding $\rho(n, n-k)$. In such cases, however, we show that it is still sometimes possible to give a useful upper bound on the dimension.

3.1 The Rank $n-3$ case

Theorem 3.1.1 *If $n \geq 8$ is not congruent to $64 \pm \epsilon$ mod 128, with $0 \leq \epsilon \leq 2$ then $\sigma(n, n, n-3) = \rho(n, n-3)$.*

The proof proceeds by first constructing spaces of the required dimensions, and then showing that no spaces of larger dimension can exist. Study of the low-dimensional cases $n < 8$ is deferred to chapter 4.

3.1.1 Lower bounds

We begin by recalling the method given in [R2] (and detailed in chapter 2) for the construction of spaces of fixed-rank matrices of large dimension.

Associate to a real vector space V (equipped with negative definite inner product) its Clifford algebra $C(V)$. Define V_1 to be the span of $1 \in C(V)$ and $V \subset C(V)$. Take W to be a n-dimensional vector space which is also a $C(V)$ module of (maximal) dimension, i.e. $\dim(V_1) = \rho(n)$. Also, $\theta(n,s)$ is defined as the minimum dimension of V_1X as X varies over all $s-$dimensional subspaces of W. We restate the following results from chapter 2.

(i) $\sigma(n,n,k) \geq \sigma(n-1,n,k) = \sigma(n,n-1,k)$ for $k < n$.
(ii) $\sigma(n,n,n-k) = \rho(n,n-k)$ for $n \geq 8$ and $0 \leq k \leq 2$.
(iii) If $X, Y \subset W$ and $U \subset V_1$ are subspaces of dimensions s, t and l respectively, such that $Y \perp UX$ then $\sigma(n-s, n-t, n-s-t) \geq l$.
(iv) $\theta(n,1) = \rho(n)$ for all n.
(v) $\theta(2n,s) \leq \theta(n,s) + s(\rho(2n) - \rho(n))$ for $s \leq n$.

There are 7 cases to consider. Assume $n \geq 8$.

Case $n \equiv 0$ (4), then $\rho(n,n-3) = \rho(n)$. Take $\dim(W) = n$, $s = 3$, $t = 0$ and $U = V_1$. Then by part (iii), we have $\sigma(n-3,n,n-3) \geq \rho(n)$ and so by part (i) $\sigma(n,n,n-3) \geq \sigma(n-3,n,n-3) \geq \rho(n)$.

Case $n \equiv 1$ (8), then $\rho(n,n-3) = \rho(n-1)$. Combine parts (i) and (ii) above to get $\sigma(n,n,n-3) \geq \sigma(n-1,n-1,n-3) \geq \rho(n-1)$.

Case $n \equiv 2$ (8), then $\rho(n,n-3) = \rho(n-2)$. As for the previous case, use parts (i) and (ii) to get $\sigma(n,n,n-3) \geq \sigma(n-2,n-2,n-3) \geq \rho(n-2)$.

Case $n \equiv 3$ (8), then $\rho(n,n-3) = \rho(n-3)$. Again, use parts (i) and (ii) to get $\sigma(n,n,n-3) \geq \sigma(n-3,n-3,n-3) \geq \rho(n-3)$.

Case $n \equiv -1$ (8), then $\rho(n,n-3) = \rho(n+1)$. Take $\dim(W) = n+1$, $s = 1$ and $t = 3$. We can take U to be V_1 providing $\theta(n+1,1) + 3 \leq n+1$, that is if $\rho(n+1) \leq n-2$ (by (iv)), which is true for $n \geq 8$. Thus $\sigma(n,n,n-3) \geq \sigma(n,n-2,n-3) \geq \rho(n+1)$.

Case $n \equiv -2$ (8), then $\rho(n, n-3) = \rho(n+2)$. Take $\dim(W) = n+2$, $s = 2$ and $t = 3$. We can take U to be V_1 if $\theta(n+2, 2) + 3 \le n+2$, that is if $\theta(n+2, 2) \le n-1$.

For $n \equiv 6$ (16) write $n + 2 = 8 + 16m$ for some $m \geq 1$. Hence we must show that $\theta(8+16m, 2) \le 5+16m$. By (v), $\theta(8+16m, 2) \le \theta(4+8m, 2)+2(\rho(8+16m)-\rho(4+8m)) \le 4 + 8m + 2(8-4) = 12 + 8m \le 5 + 16m$ for all $m \ge 1$.

For $n \equiv 14$ (16) write $n+2 = 16m$ for some $m \ge 1$. We must show that $\theta(16m, 2) \le 16m - 3$. We have $\theta(16m, 2) \le \theta(8m, 2) + 2(\rho(16m) - \rho(8m))$. Note that for any t then $\rho(2t) - \rho(t) \le 4$. Hence $\theta(16m, 2) \le 8m + 8 \le 16m - 3$ providing $m \ge 2$. For $m = 1$ then $\theta(16, 2) \le \theta(8, 2) + 2(\rho(16) - \rho(8)) = 10$ (< 13). Hence we can always take U to be V_1. Then $\sigma(n, n-1, n-3) \ge \rho(n+2)$ and so $\sigma(n, n, n-3) \ge \rho(n+2)$.

Case $n \equiv -3$ (8), then $\rho(n, n-3) = \rho(n+3)$. Take $\dim(W) = n+3$ and $s = t = 3$. To take $U = V_1$ we require $\theta(n+3, 3) + 3 \le n+3$, that is $\theta(n+3, 3) \le n$.

For $n \equiv 5$ (16) write $n+3 = 8+16m$ ($m \ge 1$), so we must show that $\theta(8+16m, 3) \le 5 + 16m$. We have $\theta(8 + 16m, 3) \le \theta(4 + 8m, 3) + 3(\rho(8 + 16m) - \rho(4 + 8m)) \le 4+8m+3(8-4) = 16+8m \le 5+16m$ providing $m \ge 2$. For $m = 1$ then consider the action of $V_1 \cong R^8$ on $R^{24} \cong R^8 \oplus R^8 \oplus R^8$. Choose a 3-dimensional subspace of R^{24} to be contained in one of the direct summands. Then V_1X is a subspace of this summand and so has dimension at most 8. Hence $\theta(24, 3) \le 8$ (< 21).

For $n \equiv 13$ (16) write $n+3 = 16m$ ($m \ge 1$). We must show that $\theta(16m, 3) \le 16m - 3$. We have $\theta(16m, 3) \le \theta(8m, 3) + 3(\rho(16m) - \rho(8m)) \le 8m + 12 \le 16m - 3$ for $m \ge 2$. For $m = 1$ then $\theta(16, 3) \le \theta(8, 3) + 3(\rho(16) - \rho(8)) = 11$ (< 13). Thus we can always take U to be V_1 and so $\sigma(n, n, n-3) \ge \rho(n+3)$.

3.1.2 Upper bounds

We now show that in almost all cases the spaces constructed above are of the largest possible dimension. Recall from chapter 2 that the existence of a $(d+1)$-dimensional space of $m \times n$ matrices, all of whose non-zero entries have fixed-rank k, gives rise to an exact sequence of real vector bundles over RP^d,

$$0 \longrightarrow G^{n-k} \longrightarrow n\epsilon \longrightarrow m\lambda \longrightarrow H^{m-k} \longrightarrow 0$$

where λ is the tautological line bundle, ϵ is the trivial line bundle, and G^{n-k}, H^{m-k} are respectively $n-k$ and $m-k$ plane bundles. Denote by gx and hx the images in $\widetilde{KO}(RP^d)$ of G^{n-k} and H^{m-k}, where x generates $\widetilde{KO}(RP^d)$ and g, h are integers.

Our approach will be to examine the different values that the function $\rho(n, n-3)$ can take. There are 6 main cases to consider, corresponding to the values of n for which $\rho(n, n-3) = 4, 8, 9, 10, 12$, or greater than 12. In each case we will assume the existence of a space of larger dimension, and analyze the corresponding exact sequence.

Proposition 3.1.2 $\rho(n, n-3) = 4 \Rightarrow \sigma(n, n, n-3) \leq 4$.

Proof If $\rho(n, n-3) = 4$ then n must be of the form $8p+4$ with $p \geq 0$. Suppose that $\sigma(8p+4, 8p+4, 8p+1) \geq 5$. We get an exact sequence over RP^4

$$0 \longrightarrow G^3 \longrightarrow (8p+4)\epsilon \longrightarrow (8p+4)\lambda \longrightarrow H^3 \longrightarrow 0$$

which splits to give the isomorphism

$$G^3 \oplus (8p+4)\lambda \cong H^3 \oplus (8p+4)\epsilon.$$

Now $\widetilde{KO}(RP^4)$ has order 8. Considering the isomorphism as an equation in K-theory, we get the following congruence

$$h - g \equiv 4 \ (8).$$

The bundles G^3, H^3 have dimension 3, so $w_4(G^3) = w_4(H^3) = 0$. For $4 \leq r \leq 7$ the class $w_4(r\lambda)$ is non-zero. Hence $g, h \in \{0, 1, 2, 3\}$. Then $h - g$ cannot take the value 4 mod 8, which gives us a contradiction.

Proposition 3.1.3 $\rho(n, n-3) = 8 \Rightarrow \sigma(n, n, n-3) \leq 8$.

Proof Here n must be of the form $16p + 8 + j$ $(-3 \leq j \leq 3)$. So suppose that $\sigma(16p+8+j, 16p+8+j, 16p+5+j) \geq 9$. The exact sequence on RP^8 is

$$0 \longrightarrow G^3 \longrightarrow (16p+8+j)\epsilon \longrightarrow (16p+8+j)\lambda \longrightarrow H^3 \longrightarrow 0$$

$\widetilde{KO}(RP^8)$ has order 16, so we get the congruence

$$(g-h) + j \equiv 8 \ (16) \qquad (-3 \leq j \leq 3).$$

The Stiefel-Whitney classes $w_4(r\lambda)$ $(4 \leq r \leq 7)$ and $w_8(r\lambda)$ $(8 \leq r \leq 15)$ are all non-zero. Thus g, h can only be $0, 1, 2$ or 3 and so $(g-h) + j \in \{0, \pm 1, \ldots, \pm 6\}$. Hence $g - h + j$ cannot be congruent to 8 mod 16.

Proposition 3.1.4 $\rho(n, n-3) = 9 \Rightarrow \sigma(n, n, n-3) \leq 9$.

Proof Here n must be of the form $32p + 16 + j$ $(-3 \leq j \leq 3)$. Assume then that $\sigma(32p+16+j, 32p+16+j, 32p+13+j) \geq 10$. The exact sequence on RP^9 is

$$0 \longrightarrow G^3 \longrightarrow (32p+16+j)\epsilon \longrightarrow (32p+16+j)\lambda \longrightarrow H^3 \longrightarrow 0.$$

The group $\widetilde{KO}(RP^9)$ has order 32. The corresponding congruence equation is

$$h - g \equiv 16 + j \ (32) \qquad (-3 \leq j \leq 3).$$

To determine which values g, h can take we analyze the stable rank 3 bundles on RP^9. Recall the following results of Adams [A2].

Theorem 3.1.5 (Adams) *The only element of $\widetilde{KO}(RP^d)$ which can be represented by a Spin(3) bundle is* 0. *The only elements which can be represented by Spin(4) bundles are* 0 *and* $4x$. *For $d \leq 8$ and $d \geq 13$, the only element which can be represented by a Spin(5) bundle with $w_4 \neq 0$ is $4x$. For $d = 9, 10$ or 11 we also have $-12x$, and for $d = 12$ we obtain also $-60x$.*

Let M^3 be an arbitrary real 3-plane bundle on RP^9. Write $M^3 \sim mx$ to mean that the bundle M^3 corresponds to the element mx $(= m(\lambda - \epsilon))$ in $\widetilde{KO}(RP^9)$ and so to the element $mx + 3\epsilon$ in $KO(RP^9)$. In this case, we have $m \in \{0, 1, \ldots, 31\}$. Adams' theorem will be used to show that m can only be $0, 1, 2, 3, 17$ or 18.

Case $m \equiv 0 \ (4)$: Then $w_1(M^3) = w_2(M^3) = 0$, so M^3 has a Spin(3) structure. By the theorem, the only such element is 0.

Case $m \equiv 1 \ (8)$: In $KO(RP^9)$ we have $M^3 \sim m(\lambda - \epsilon) + 3\epsilon$. Tensoring by λ gives

$$M^3 \otimes \lambda \sim m(\epsilon - \lambda) + 3\lambda = (3 - m)(\lambda - \epsilon) + 3\epsilon.$$

Then $M^3 \otimes \lambda \sim (3-m)x$ in $\widetilde{KO}(RP^9)$ and so $(M^3 \otimes \lambda) \oplus 2\lambda \sim (5-m)x$. Now $5 - m \equiv 0 \ (4)$ so $(M^3 \otimes \lambda) \oplus 2\lambda$ has a Spin(5) structure. Also, $w_4((M^3 \otimes \lambda) \oplus 2\lambda) = w_4((5-m)\lambda)$ which is non-zero since $m \equiv 1 \ (8)$. So by the theorem, $5 - m \equiv 4$ or $-12 \ (32)$, i.e. $m \equiv 1$ or $17 \ (32)$.

Case $m \equiv 2 \ (8)$: Here $M^3 \oplus 2\lambda \sim (m+2)x$ and $m + 2 \equiv 4 \ (8)$. Hence $M^3 \oplus 2\lambda$ has a Spin(5) structure. The class $w_4(M^3 \oplus 2\lambda)$ is non-zero. We must have $m + 2 \equiv 4$ or $-12 \ (32)$, i.e. $m \equiv 2$ or $18 \ (32)$.

Case $m \equiv 3$ (4) : As before, we can write $M^3 \otimes \lambda \sim (3-m)x$. Then $M^3 \otimes \lambda$ has a Spin(3) structure, giving $3 - m = 0$, or $m = 3$.

Case $m \equiv 5$ (8) : Write $m = 5 + 8p$. Then $w_5(M^3) = w_5((5+8p)\lambda) \neq 0$. But M^3 has rank 3, so $w_5(M^3)$ must vanish. This gives a contradiction.

Case $m \equiv 6$ (8) : Here $w_6(M^3) \neq 0$.

So the only possibilities for g, h are $0, 1, 2, 3, 17$ and 18. In fact, we can also rule out 17 and 18: According to [LR1], the maximum number of linearly independent sections of 17λ on RP^9 is 12. Hence the geometric dimension of 17λ on RP^9 (and thus of the element $17x$ in $\widetilde{KO}(RP^9)$) is $17 - 12 = 5$. Also, the geometric dimension of 18λ on RP^9 is 6. So $g, h \in \{0, 1, 2, 3\}$ and the congruence $h - g \equiv 16 + j$ (32) $(0 \leq j \leq 3)$ cannot be satisfied.

Proposition 3.1.6 $\rho(n, n-3) = 10 \Rightarrow \sigma(n, n, n-3) \leq 10$.

Proof Here n must be of the form $64p + 32 + j$ $(-3 \leq j \leq 3)$. Assume then that $\sigma(64p + 32 + j, 64p + 32 + j, 64p + 29 + j) \geq 11$. The exact sequence on RP^{10} is

$$0 \longrightarrow G^3 \longrightarrow (64p + 32 + j)\epsilon \longrightarrow (64p + 32 + j)\lambda \longrightarrow H^3 \longrightarrow 0.$$

The group $\widetilde{KO}(RP^{10})$ has order 64. The corresponding congruence equation is

$$h - g \equiv 32 + j \ (64) \qquad (-3 \leq j \leq 3).$$

The analysis of stable rank 3 bundles on RP^{10} is similar to that on RP^9. The only difference arises from considering the case $m \equiv 2$ (8). We have the equation $m + 2 = 4$ or -12 as before, but here we are working mod 64 instead of mod 32. Hence $m = 2$ or 50. Thus every rank 3 bundle on RP^{10} corresponds to an element mx in $\widetilde{KO}(RP^{10})$, with $m \in \{0, 1, 2, 3, 17, 50\}$. On RP^{10}, 17λ has geometric dimension 5 (by [LR1]).

We use γ operations (see [A6]) to show that 50λ must have geometric dimension at least 4 on RP^{10}. For otherwise, the polynomial $(1 + xt)^{50}$ (in the indeterminate t and with coefficients in $\widetilde{KO}(RP^{10})$) would have degree ≤ 3. In particular, the coefficient of t^4 would be 0. That is,

$$\binom{50}{4} x^4 = 0.$$

The ring structure gives $x^4 = -8x$, so

$$\binom{50}{4}(-8) \equiv 0 \ (64).$$

which, after simplifying, gives the contradiction $1 \equiv 0 \ (2)$.

Hence $g, h \in \{0, 1, 2, 3\}$ and the congruence $h - g \equiv 32 + j \ (64)$ cannot be satisfied.

Proposition 3.1.7 $\rho(n, n-3) = 12$ *and* $n \equiv \pm 3 \ (64) \Rightarrow \sigma(n, n, n-3) \leq 12$.

Proof Here n must be of the form $128p + 64 + j$ with $j = \pm 3$. As usual, assume that $\sigma(128p + 64 + j, 128p + 64 + j, 128p + 61 + j) \geq 13$. The exact sequence on RP^{12} is

$$0 \longrightarrow G^3 \longrightarrow (128p + 64 + j)\epsilon \longrightarrow (128p + 64 + j)\lambda \longrightarrow H^3 \longrightarrow 0.$$

The group $\widetilde{KO}(RP^{12})$ has order 128. The corresponding congruence is

$$h - g \equiv 64 + j \ (128) \qquad (j = \pm 3).$$

A similar analysis to the previous examples shows that every rank 3 bundle on RP^{12} must correspond to mx in the reduced K-theory, with $m \in \{0, 1, 2, 3, 17, 65, 66, 114\}$. By [LR1], the geometric dimension of 17λ on RP^{12} is 8, so $m \neq 17$. Also, another calculation with γ operations shows that $114x$ cannot have geometric dimension ≤ 3. The cases $j = \pm 3$ for the congruence $h - g \equiv 64 + j \ (128)$ yield contradictions.

Proposition 3.1.8 $\rho(n, n-3) > 12 \Rightarrow \sigma(n, n, n-3) \leq \rho(n, n-3)$.

Proof Since $\sigma(n, n, n-3) \geq \rho(n, n-3)$ and the next value (i.e. larger than 12) that $\rho(n, n-3)$ can take is 16, we may certainly assume that $\sigma(n, n, n-3) \geq 14$. Write $d + 1 = \sigma(n, n, n-3)$, so there is an exact sequence of vector bundles on RP^d and a corresponding congruence, which we may write as

$$n + g - h \equiv 0 \ (2^{\phi(d)}).$$

We can deal with the case $\rho(n, n-3) = 16$, and all subsequent cases, by making use of another result of Adams [A2].

Lemma 3.1.9 (Adams) *For $d \geq 13$, the only elements of geometric dimension ≤ 3 in $\widetilde{KO}(RP^d)$ are the elements rx with $0 \leq r \leq 3$.*

This tells us that $g, h \in \{0, 1, 2, 3\}$.

Recall lemma 2.4.7, which says that if $v_2(n)$ is the exponent of the highest power of 2 dividing n then $v_2(n) = \phi(\rho(n)) - 1$.

For any positive $t \equiv 0 \ (2^{\phi(d)})$ we have $v_2(t) \geq \phi(d)$. By lemma 2.4.7, we must have $\phi(\rho(t)) - 1 \geq \phi(d)$, or $\phi(\rho(t)) - \phi(d) \geq 1$, giving $\rho(t) \geq d + 1 = \sigma(n, n, n-3)$. In this case $t = n + g - h$, which takes values $n \pm \epsilon \ (0 \leq \epsilon \leq 3)$, thus $\sigma(n, n, n-3) \leq \rho(n, n-3)$.

This completes the proof of the Theorem 3.1.1.

3.2 The Rank $n-4$ case

Theorem 3.2.1 *Suppose $n \geq 12$ satisfies one of the conditions:*
(i) $\rho(n, n-4) = 8, 9$ or 10.
(ii) $\rho(n, n-4) = 12$ and n is not congruent to $64 \pm \epsilon, \ 0 \leq \epsilon \leq 3 \ mod \ 128$.
(iii) $\rho(n, n-4) > 12$ and congruent to 0 mod 8.
Then $\sigma(n, n, n-4) = \rho(n, n-4)$.

The proof proceeds as for the rank $n-3$ case: spaces of dimension $\rho(n, n-4)$ are constructed for all $n \geq 12$. (The cases $n < 12$ are deferred to chapter 4.) We then show that in most cases, spaces of higher dimension cannot exist.

3.2.1 Lower bounds

Assume $n \geq 12$. There are 9 cases.

Case $n \equiv 0 \ (8)$, then $\rho(n, n-4) = \rho(n)$. Take $\dim(W) = n$, $s = 4$, $t = 0$ and $U = V_1$. Then $\sigma(n-4, n, n-4) \geq \rho(n)$ and so $\sigma(n, n, n-4) \geq \rho(n)$.

Case $n \equiv 1 \ (8)$, then $\rho(n, n-4) = \rho(n-1)$. We can now use theorem 3.1.1 to get $\sigma(n, n, n-4) \geq \sigma(n-1, n-1, n-4) \geq \rho(n-1)$.

Case $n \equiv 2 \ (8)$, then $\rho(n, n-4) = \rho(n-2)$. We use theorem 2.4.6 (iii) to get $\sigma(n, n, n-4) \geq \sigma(n-2, n-2, n-4) \geq \rho(n-2)$.

Case $n \equiv 3 \ (8)$, then $\rho(n, n-4) = \rho(n-3)$. We use theorem 2.4.6 (ii) to get: $\sigma(n, n, n-4) \geq \sigma(n-3, n-3, n-4) \geq \rho(n-3)$.

Case $n \equiv 4 \ (16)$, then $\rho(n, n-4) = \rho(n-4)$. Here we just use theorem 2.4.6 (i): $\sigma(n, n, n-4) \geq \sigma(n-4, n-4, n-4) \geq \rho(n-4)$.

Case $n \equiv -1$ (8), then $\rho(n, n-4) = \rho(n+1)$. Take $\dim(W) = n+1$, $s = 1$ and $t = 4$. We can take U to be V_1 providing $\theta(n+1, 1) + 4 \leq n+1$, that is if $\rho(n+1) \leq n-3$, which is true for $n \geq 8$. Then $\sigma(n, n, n-4) \geq \sigma(n, n-3, n-4) \geq \rho(n+1)$.

Case $n \equiv -2$ (8), then $\rho(n, n-4) = \rho(n+2)$. Take $\dim(W) = n+2$, $s = 2$ and $t = 4$. We can take $U = V_1$ providing $\theta(n+2, 2)+4 \leq n+2$, that is if $\theta(n+2, 2) \leq n-2$. Recall the upper bounds for $\theta(n+2, 2)$ with $n \equiv -2$ (8) determined in the previous section: for $n+2 = 8+16m$, then $\theta(8+16m, 2) \leq 12+8m$; for $m > 1$ and $n+2 = 16m$ then $\theta(16m, 2) \leq 8m+8$; $\theta(16, 2) \leq 10$. Then $\sigma(n, n, n-4) \geq \sigma(n, n-2, n-4) \geq \rho(n+2)$.

Case $n \equiv -3$ (8), then $\rho(n, n-4) = \rho(n+3)$. Take $\dim(W) = n+3$, $s = 3$ and $t = 4$. We can take $U = V_1$ providing $\theta(n+3, 3)+4 \leq n+3$, that is if $\theta(n+3, 3) \leq n-1$. Again, it turns out that the upper bounds for $\theta(n+3, 3)$ proved earlier suffice for this case: for $n+3 = 8+16m$, $\theta(8+16m, 3) \leq 16+8m$ and $\theta(24, 3) \leq 8$. For $n+3 = 16m$ and $n > 13$, $\theta(16m, 3) \leq 8m+12$, and $\theta(16, 3) \leq 11$. Thus $\sigma(n, n, n-4) \geq \sigma(n, n-1, n-4) \geq \rho(n+3)$.

Case $n \equiv -4$ (16), then $\rho(n, n-4) = \rho(n+4)$. Take $\dim(W) = n+4$ and $s = t = 4$. We can take U to be V_1 providing $\theta(n+4, 4) + 4 \leq n+4$, that is if $\theta(n+4, 4) \leq n$. Write $n+4 = 16m$ for some $m \geq 1$. We have $\theta(n+4, 4) = \theta(16m, 4) \leq \theta(8m, 4) + 4(\rho(16m) - \rho(8m)) \leq 8m+16 \leq 16m-4$ providing $m \geq 3$. For $m = 1$ then $\theta(16, 4) \leq \theta(8, 4)+4(\rho(16)-\rho(8)) = 12$ and for $m = 2$, $\theta(32, 4) \leq \theta(16, 4)+4(\rho(32)-\rho(16)) \leq 20$. Then $\sigma(n, n, n-4) \geq \rho(n+4)$.

3.2.2 Upper bounds

Proposition 3.2.2 $\rho(n, n-4) = 8 \Rightarrow \sigma(n, n, n-4) \leq 8$.

Proof By inspection, n must be of the form $16p+8+j$ $(-3 \leq j \leq 3)$. So suppose that $\sigma(16p+8+j, 16p+8+j, 16p+4+j) \geq 9$. The exact sequence on RP^8 is

$$0 \longrightarrow G^4 \longrightarrow (16p+8+j)\epsilon \longrightarrow (16p+8+j)\lambda \longrightarrow H^4 \longrightarrow 0.$$

The congruence we get from this is

$$h - g \equiv 8 + j \ (16) \qquad (-3 \leq j \leq 3).$$

Now $w_4(r\lambda) \neq 0$ $(5 \leq r \leq 7)$ and $w_8(r\lambda) \neq 0$ $(8 \leq r \leq 15)$. Thus $g, h \in \{0, 1, 2, 3, 4\}$. Then $h - g \in \{0, 1, 2, 3, 4, 12, 13, 14, 15\}$, and $8 + j \in \{5, 6, 7, 8, 9, 10, 11\}$. This gives a contradiction.

Proposition 3.2.3 $\rho(n, n-4) = 9 \Rightarrow \sigma(n, n, n-4) \leq 9$.

Proof Here n must be of the form $32p + 16 + j$ $(-4 \leq j \leq 4)$. So suppose that $\sigma(32p + 16 + j, 32p + 16 + j, 32p + 12 + j) \geq 10$. The exact sequence on RP^9 is

$$0 \longrightarrow G^4 \longrightarrow (32p + 16 + j)\epsilon \longrightarrow (32p + 16 + j)\lambda \longrightarrow H^4 \longrightarrow 0,$$

which gives

$$h - g \equiv 16 + j\ (32) \qquad (-4 \leq j \leq 4).$$

We now analyze the stable rank 4 real bundles on RP^9. We will again appeal to Adams' results. So let M^4 be an arbitrary real 4-plane bundle over RP^9, whose image in $KO(RP^9)$ is the element $mx + 4\epsilon$ for some $m \in \{0, 1, \ldots, 31\}$.

Case $m \equiv 0\ (4)$: M^4 has a Spin(4) structure; theorem 3.1.5 gives $m = 0$ or 4.

Case $m \equiv 1\ (8)$: In $KO(RP^9)$ we have $M^4 \sim m(\lambda - \epsilon) + 4\epsilon$. Then

$$M^4 \otimes \lambda \sim m(\epsilon - \lambda) + 4\lambda = (4 - m)(\lambda - \epsilon) + 4\epsilon.$$

That is, $M^4 \otimes \lambda \sim (4 - m)x$ in $\widetilde{KO}(RP^9)$ and so $(M^4 \otimes \lambda) \oplus \lambda \sim (5 - m)x$. Now $5 - m \equiv 4\ (8)$. Thus $(M^4 \otimes \lambda) \oplus \lambda$ has a Spin(5) structure. Also, $w_4((M^4 \otimes \lambda) \oplus \lambda) \neq 0$. We must have $(5 - m)x = 4x$ or $-12x$, so $m = 1$ or 17.

Case $m \equiv 2\ (8)$: Write $m = 2 + 8p$. The class $w_8((2 + 8p)\lambda)$ is non-zero unless p is even: $p = 0$ gives the element $2x$; $p = 2$ gives $18x$. These are the only possibilities.

Case $m \equiv 3\ (8)$: Write $m = 3 + 8p$ for some $p \geq 0$. Then $M^4 \oplus \lambda \sim (4 + 8p)x$. So $M^4 \oplus \lambda$ is Spin(5), with $w_4 \neq 0$. Then $(m + 1)x = 4x$ or $-12x$, i.e. $m = 3$ or 19.

Case $m \equiv 5\ (8)$: We have $w_5(M^4) \neq 0$, which is a contradiction.

Case $m \equiv 6\ (8)$: $w_6(M^4) \neq 0$.

Case $m \equiv 7\ (8)$: $w_5(M^4) \neq 0$.

Hence any 4-plane bundle on RP^9 must correspond to mx in the reduced K-theory with $m \in \{0, 1, 2, 3, 4, 17, 18, 19\}$. We know from the previous section that on RP^9, 17λ and 18λ respectively have geometric dimensions 5 and 6. Also from [LR1], we see that 19λ

has geometric dimension 6 on RP^9. So the only elements in $\widetilde{KO}(RP^9)$ of geometric dimension ≤ 4 are the elements mx with $0 \leq m \leq 4$. Working mod 32, we see that $h-g$ can take values in $\{0, \pm 1, \ldots, \pm 4\}$, while $12 \leq 16+j \leq 20$. Hence the congruence cannot be satisfied.

Proposition 3.2.4 $\rho(n, n-4) = 10 \Rightarrow \sigma(n, n, n-4) \leq 10$.

Proof Here n must be of the form $64p + 32 + j$ $(-4 \leq j \leq 4)$. So suppose that $\sigma(64p + 32 + j, 64p + 32 + j, 64p + 28 + j) \geq 11$. The exact sequence on RP^{10} is

$$0 \longrightarrow G^4 \longrightarrow (64p + 32 + j)\epsilon \longrightarrow (64p + 32 + j)\lambda \longrightarrow H^4 \longrightarrow 0.$$

which gives the congruence

$$h - g \equiv 32 + j \ (64) \qquad (-4 \leq j \leq 4).$$

When considering the stable rank 4 bundles on RP^{10}, the calculations for the cases $m \equiv 0, 1, 4, 5, 6, 7$ (8) are identical to those for RP^9. For $m \equiv 2$ (8), we get the elements $2x, 18x, 34x$ and $50x$. For $m \equiv 3$ (8), we get $3x$ and $51x$. So any rank 4 bundle on RP^{10} must have image $mx + 4\epsilon$ in K-theory with $m \in \{0, 1, 2, 3, 4, 17, 18, 34, 50, 51\}$.

We know from [LR1] that $17\lambda, 18\lambda$ respectively have geometric dimensions 5 and 6 on RP^{10}. According to [LR2], 34λ has geometric dimension 6 over RP^{10}. There are no relevant tables for the cases $50x$ and $51x$, and γ operations do not prove that their geometric dimension exceeds 4. Instead, we use the following lemma.

Lemma 3.2.5 *In $\widetilde{KO}(RP^n)$ the element mx has geometric dimension $\leq r$ if and only if the element $(r-m)x$ has geometric dimension $\leq r$.*

Proof Suppose that mx has geometric dimension $\leq r$. Then there is some r-plane bundle η on RP^n such that

$$\eta \oplus (m-r)\epsilon \cong m\lambda.$$

In $KO(RP^n)$ we have the correspondence $\eta \sim mx + r\epsilon = m(\lambda - \epsilon) + r\epsilon$. Tensoring by the line bundle λ, and using the fact that $\lambda \otimes \lambda \cong \epsilon$ over RP^n gives

$$\eta \otimes \lambda \sim m(\epsilon - \lambda) + r\lambda = (r-m)(\lambda - \epsilon) + r\epsilon.$$

Therefore the element $(r-m)x$ is represented by the rank r bundle $\eta \otimes \lambda$ and so has geometric dimension $\leq r$. The argument works in the opposite direction.

Applying the lemma to the cases above, we see that $50x$ has geometric dimension ≤ 4 in $\widetilde{KO}(RP^{10})$ if and only if $(4-50)x = -46x = 18x$ has geometric dimension ≤ 4. Similarly, $51x$ has geometric dimension ≤ 4 if and only if $(4-51)x = -47x = 17x$ has geometric dimension ≤ 4. But $17\lambda, 18\lambda$ have geometric dimension greater than 4 on RP^{10}, so we have a contradiction.

So the only elements of geometric dimension ≤ 4 in $\widetilde{KO}(RP^{10})$ are the elements mx with $0 \leq m \leq 4$. Hence the congruence $h - g \equiv 32 + j$ (64) cannot be satisfied.

Proposition 3.2.6 *$\rho(n, n-4) = 12$ and $n \equiv \pm 4$ (64) $\Rightarrow \sigma(n, n, n-4) \leq 12$.*

Proof Here n must be of the form $128p + 64 + j$ with $j = \pm 4$. So suppose that $\sigma(128p + 64 + j, 128p + 64 + j, 128p + 60 + j) \geq 13$. The exact sequence on RP^{12} is

$$0 \longrightarrow G^4 \longrightarrow (128p + 64 + j)\epsilon \longrightarrow (128p + 64 + j)\lambda \longrightarrow H^4 \longrightarrow 0.$$

and the corresponding congruence is

$$h - g \equiv 64 + j \ (128) \qquad (j = \pm 4).$$

To analyze stable 4-plane bundles over RP^{12}, we use (in addition to theorem 3.1.5) a result of Lam and Randall [LR3] on Spin(6) bundles over RP^d.

Lemma 3.2.7 (Lam and Randall) *Let $d \geq 12$ and suppose that $4k\lambda$ has geometric dimension ≤ 6 on RP^d. Then*
(i) $16k \equiv 0$ $(2^{[d/2]})$ if k is even.
(ii) $4k \equiv 4$ $(2^{[d/2]})$ if k is odd.

This is useful for the case $m \equiv 2$ (8): write $m = 2 + 8p$, so $M^4 \oplus 2\lambda \sim 4(1+2p)x$. Then by part (ii) of the lemma, $4(1+2p) \equiv 4$ (2^6), i.e. $8p \equiv 0$ (64). Thus $m = 2$ or 66.

Only the elements mx with $m \in \{0, 1, 2, 3, 4, 17, 65, 66, 67, 115\}$ can have geometric dimension ≤ 4 in $\widetilde{KO}(RP^{12})$. From the previous section, we know that 17λ has geometric dimension 8 on RP^{12}, and γ operations show that 115λ cannot have geometric dimension ≤ 4. Unfortunately we have been unable to rule out the possibility of one of 65λ, 66λ or 67λ being stably isomorphic to some 4-plane bundle. But for $j = \pm 4$ the congruence cannot be satisfied.

Proposition 3.2.8 *$\rho(n, n-4) > 12$ and congruent to 0 (8) $\Rightarrow \sigma(n, n, n-4) \leq \rho(n, n-4)$.*

Proof Let $\sigma(n, n, n-4) = d+1$, where $\rho(n, n-4) > 12$. The corresponding exact sequence over RP^d implies the congruence

$$n + g - h \equiv 0 \ (2^{\phi(d)}).$$

Since $\rho(n, n-4)$ must be ≥ 16, d is large enough to apply lemma 3.2.7. It turns out that only the elements $0, x, 2x, 3x, 4x$ and $(2 + 2^{\phi(d)-1})x$ in $\widetilde{KO}(RP^d)$ can have geometric dimension ≤ 4.

The element $(2+2^{\phi(d)-1})x$ arises from considering those elements mx with $m \equiv 2$ (8). If $d \equiv 0$ (8) then this element cannot have geometric dimension ≤ 4: let $M^4 \sim (2+8p)x$. Then $M^4 \oplus 2\lambda \sim 4(1+2p)x$ and by the lemma, $8p \equiv 0$ $(2^{[d/2]})$.

The table below compares the functions $[d/2]$ and $\phi(d)$.

d	$8q$	$8q+1$	$8q+2$	$8q+3$	$8q+4$	$8q+5$	$8q+6$	$8q+7$
$[d/2]$	$4q$	$4q$	$4q+1$	$4q+1$	$4q+2$	$4q+2$	$4q+3$	$4q+3$
$\phi(d)$	$4q$	$4q+1$	$4q+2$	$4q+2$	$4q+3$	$4q+3$	$4q+3$	$4q+3$

For $d \equiv 0, 6$ or 7 (8) we have $[d/2] = \phi(d)$. Thus $8p \equiv 0$ $(2^{\phi(d)})$ and so $m = 2$.

For $d \equiv 1, 2, 3, 4$ or 5 (8) we get $[d/2] = \phi(d) - 1$ and so $m = 2 + 8p = 2 + 2^{\phi(d)-1}p'$. Then $p' = 0$ gives $m = 2$ and $p' = 1$ gives $m = 2 + 2^{\phi(d)-1}$. These are the only possibilities, since $p' = 2$ gives $m = 2 + 2^{\phi(d)} = 2$.

The function $\rho(n, n-4)$ is congruent to $0, 1, 2$ or 4 mod 8. For $\rho(n, n-4) \equiv 0$ (8) (and larger than 12) then $g, h \in \{0, 1, 2, 3, 4\}$. By a similar argument to the rank $n-3$ case, we must have $\sigma(n, n, n-4) \leq \rho(n, n-4)$.

This completes the proof of Theorem 3.2.1.

The proof of lemma 3.2.7 uses the representations of the Lie group $\mathrm{Spin}^c(n)$. A similar approach can be used to study the geometric dimension of $(4k+2)x$ in $\widetilde{KO}(RP^d)$. Specifically, one considers the representations of $\mathrm{Spin}^c(4)$. The following result avoids any conditions on d, but does not say anything new about the outstanding cases.

Proposition 3.2.9 *Suppose that the bundle $(4k+2)\lambda$ has geometric dimension ≤ 4 over RP^d. Then $4k \equiv 0$ mod $2^{[d/2]}$.*

Proof See Appendix A.

Some remarks on outstanding cases

To remove the clumsy condition 'n not congruent to $64 \pm \epsilon$ mod 128 for $0 \leq \epsilon \leq 2$' in the statement of theorem 3.1.1, it is sufficient to show that the geometric dimension of 65λ and 66λ on RP^{12} must both be larger than 3.

In $\widetilde{KO}(RP^{12})$, we have

$$g.dim(66x) \leq g.dim(65x) + g.dim(x) = g.dim(65x) + 1,$$

and so $g.dim(65x) \geq g.dim(66x) - 1$. If one could show that $66x$ has geometric dimension at least 5 then we would automatically have that the geometric dimension of $65x$ is at least 4. In fact one can say more: applying lemma 3.2.5 gives

$$\begin{aligned} g.dim(65x) \leq 3 &\iff g.dim((3-65)x) \leq 3 \\ &\iff g.dim(-62x) \leq 3 \\ &\iff g.dim(66x) \leq 3. \end{aligned}$$

So to prove $\sigma(n, n, n-3) \leq \rho(n, n-3)$ for all n, it is sufficient to show that just one of 65λ, 66λ has geometric dimension at least 4 on RP^{12}.

To prove part (ii) of theorem 3.2.1 for all $n \geq 12$, one must show that each of 65λ, 66λ and 67λ have geometric dimension at least 5 on RP^{12}. Again, one can use lemma 3.2.5 to slightly improve the situation.

$$\begin{aligned} g.dim(65x) \leq 4 &\iff g.dim((4-65)x) \leq 4 \\ &\iff g.dim(-61x) \leq 4 \\ &\iff g.dim(67x) \leq 4. \end{aligned}$$

So it is enough to show that over RP^{12}, 66λ and at least one of 65λ, 67λ have geometric dimension exceeding 4. To prove $\sigma(n, n, n-4) \leq \rho(n, n-4)$ for all n, one must also show that $(2+2^{\phi(n)-1})\lambda$ has geometric dimension larger than 4 for $\rho(n, n-4) \equiv 1, 2$ or 4 mod 8. (The case $\rho(n, n-4) \equiv 0$ mod 8 is proved, and these are the only possibilities.)

A similar situation has been studied - from the viewpoint of *normed bilinear maps* - by Lam and Yiu [LY1]. Their results are related to spaces of rectangular matrices (specifically, to $\sigma(n, n-i, n-i)$ for $i \leq 5$). Some outstanding cases occur, which are related to our ignorance on the geometric dimensions of 65λ, 66λ and 67λ on RP^{12}.

3.3 The general case

In this section we discuss the limitations of the hypothesis that if n is sufficiently large then $\sigma(n,n,n-k) = \rho(n,n-k)$. We show that for every k and for n sufficiently large then $\rho(n,n-k)$ is always a lower bound for $\sigma(n,n,n-k)$. For $5 \leq k \leq 7$ there are some cases where equality occurs, but for $k = 8$ our approach yields no further results and for $k = 9$ there are infinitely many n for which we can construct spaces of $n \times n$ matrices of rank $n-9$ whose dimension exceeds $\rho(n,n-9)$.

The final result of this chapter demonstrates that for some large-rank spaces - where K-theory does not give any information - a Stiefel-Whitney class argument can provide useful (though usually weaker) upper bounds.

Theorem 3.3.1 *For any positive integer k, there exists an integer N_k such that for every $n \geq N_k$, then $\sigma(n,n,n-k) \geq \rho(n,n-k)$.*

Proof. By induction on k.

For $k = 1$ take $N_1 = 8$, so $n \geq N_1 \Rightarrow \sigma(n,n,n-1) \geq \rho(n,n-1)$ by theorem 2.4.6(ii).

Let $k = l-1$ for some fixed $l \geq 2$ and assume that there exists an N_{l-1} such that

$$n \geq N_{l-1} \Rightarrow \sigma(n,n,n-l+1) \geq \rho(n,n-l+1).$$

Now let $k = l$. We must find an N_l such that

$$n \geq N_l \Rightarrow \sigma(n,n,n-l) \geq \rho(n,n-l).$$

We have

$$\begin{aligned}\rho(n,n-l) &= \max\{\rho(n-l),\ldots,\rho(n),\ldots,\rho(n+l)\}\\ &= \max\{\rho(n-1,n-l),\rho(n+l-1),\rho(n+l)\}.\end{aligned}$$

There are 3 cases.

Case $\rho(n,n-l) = \rho(n-1,n-l)$: Use $\sigma(n,n,n-l) \geq \sigma(n-1,n-1,n-l) \geq \rho(n-1,n-l)$ for $n-1 \geq N_{l-1}$ by the inductive hypothesis. So choose $N_{l_0} = N_{l-1}+1$.

Case $\rho(n,n-l) = \rho(n+l)$. In the notation of section 2.4, take $\dim(W) = n+l$ and $s = t = l$. One can take U to be V_1 providing $l + \theta(n+l,l) \leq n+l$.

That is, we need to find an N_{l_1} such that

$$\rho(n, n-l) = \rho(n+l) \text{ and } n \geq N_{l_1} \;\Rightarrow \theta(n+l, l) \leq n.$$

For convenience, we restate some properties of the functions θ and ρ:

(i) $\theta(2n, s) \leq \theta(n, s) + s(\rho(2n) - \rho(n))$ for all $s \leq n$.
(ii) $\theta(n, s) \leq n$ for all $s \leq n$.
(iii) $\rho(2n) - \rho(n) \leq 4$ for all n.

Clearly $n+l$ must be even. If $n \geq l$ then

$$\begin{aligned}
\theta(n+l, l) &\leq \theta(\tfrac{n+l}{2}, l) + l(\rho(n+l) - \rho(\tfrac{n+l}{2})) && \text{(by (i))}\\
&\leq \tfrac{n+l}{2} + l(\rho(n+l) - \rho(\tfrac{n+l}{2})) && \text{(by (ii))}\\
&\leq \tfrac{n+l}{2} + 4l && \text{(by (iii))}
\end{aligned}$$

Now $\frac{n+l}{2} + 4l \leq n$ providing $n \geq 9l$. So taking $N_{l_1} = 9l$ allows us to take U to be V_1 and so for $n \geq N_{l_1}$ with $\rho(n, n-l) = \rho(n+l)$, $\sigma(n, n, n-l) \geq \rho(n+l)$.

Case $\rho(n, n-l) = \rho(n+l-1)$. Take $\dim(W) = n+l-1,\ s = l,\ t = l-1$. Then U can be taken to be V_1 providing $\theta(n+l-1, l) + l - 1 \leq n+l-1$. That is, if

$$\theta(n+l-1, l) \leq n$$

which, by a similar argument to that above, is true for $n \geq 9l - 1 =: N_{l_2}$. Then for all $n \geq N_{l_2}$ (with $\rho(n, n-l) = \rho(n-l+1)$) we can take $U = V_1$ and so $\sigma(n, n, n-l) \geq \sigma(n-1, n, n-l) \geq \rho(n+l-1)$.

Since $N_{l_1} > N_{l_2}$, we can complete the induction by setting

$$N_l = \max\{N_{l_0}, N_{l_1}\} = \max\{N_{l-1}+1, 9l\}.$$

Then for all $n \geq N_l$ we have $\sigma(n, n, n-l) \geq \rho(n, n-l)$. This completes the proof.

Remark. The 'N_l' found in theorem 3.3.1 is not best possible. But it does tell us that for each k there can only be a finite number of exceptions to the lower bound $\sigma(n, n, k) \geq \rho(n, n-k)$.

Theorem 3.3.2 *For $n \geq 12$*
(i) $\rho(n, n-5) = 8 \Rightarrow \sigma(n, n, n-5) = 8$.
(ii) $\rho(n, n-6) = 8 \Rightarrow \sigma(n, n, n-6) = 8$.
(iii) $\rho(n, n-7) = 8 \Rightarrow \sigma(n, n, n-7) = 8$.
(iv) $\rho(n, n-5) = 9$ *and* $n \equiv 11$ *or* 21 *mod* $32 \Rightarrow \sigma(n, n, n-5) = 9$.

Proof

(i) $\rho(n, n-5) = 8 \Rightarrow n$ is of the form $16p + 8 + j$ $(-2 \leq j \leq 2)$. Assume that $\sigma(16p+8+j, 16p+8+j, 16p+3+j) \geq 9$. The exact sequence on RP^8 is

$$0 \longrightarrow G^5 \longrightarrow (16p+8+j)\epsilon \longrightarrow (16p+8+j)\lambda \longrightarrow H^5 \longrightarrow 0$$

which gives the congruence

$$h - g \equiv 8 + j \ (16) \qquad (-2 \leq j \leq 2).$$

Now $8 + j \in \{6, 7, 8, 9, 10\}$. But $w_6(6\lambda) \neq 0$, $w_7(7\lambda) \neq 0$ and $w_8(r\lambda) \neq 0$ $(8 \leq r \leq 15)$, so $g, h \in \{0, 1, \ldots, 5\}$, $h - g \in \{0, 1, \ldots, 5, 11, 12, \ldots, 15\}$ and the congruence cannot be satisfied.

We know from theorem 3.3.1 that for all n sufficiently large then spaces of the required dimensions exist. We give explicit constructions for the cases covered by the theorem.

For $n \equiv 8$ (16), then $\rho(n, n-5) = \rho(n)$. Take dim(W)$= n$, $s = 5$, $t = 0$ and $U = V_1$. Then $\sigma(n, n, n-5) \geq \sigma(n-5, n, n-5) \geq \rho(n)$.

For $n \equiv 9$ (16), then $\rho(n, n-5) = \rho(n-1)$. Then $\sigma(n, n, n-5) \geq \sigma(n-1, n-1, n-5) \geq \rho(n-1)$ by theorem 3.2.1.

For $n \equiv 10$ (16), then $\rho(n, n-5) = \rho(n-2)$. Then $\sigma(n, n, n-5) \geq \sigma(n-2, n-2, n-5) \geq \rho(n-2)$ by theorem 3.1.1.

For $n \equiv 7$ (16), then $\rho(n, n-5) = \rho(n+1)$. Take dim(W)$= n+1$, $s = 1$, $t = 5$. We can take U to be V_1 providing $\theta(n+1, 1) + 5 \leq n+1$, that is if $\rho(n+1) \leq n-4$, which is true for $n \geq 8$. Then $\sigma(n, n, n-5) \geq \sigma(n, n-4, n-5) \geq \rho(n+1)$.

For $n \equiv 6$ (16), then $\rho(n, n-5) = \rho(n+2)$. Take dim(W)$= n+2$, $s = 2$, $t = 5$. We can take $U = V_1$ providing $\theta(n+2, 2) + 5 \leq n+2$, that is if $\theta(n+2, 2) \leq n-3$. Write $n+2 = 8+16m$ $(m \geq 1)$. The upper bounds $\theta(8+16m, 2) \leq 12+8m$ and $\theta(24, 2) \leq 8$ obtained in 3.1.1 are sufficient. Then $\sigma(n, n, n-5) \geq \sigma(n, n-3, n-5) \geq \rho(n+2)$.

(ii) $\rho(n, n-6) = 8 \iff n = 16p + 8 + j$ $(-1 \leq j \leq 1)$. The assumption of a space of dimension ≥ 9 leads again to the congruence $h - g \equiv 8 + j$ (16) for $-1 \leq j \leq 1$. Here $8 + j \in \{7, 8, 9\}$ and $g, h \in \{0, 1, \ldots, 6\}$, so $h - g \in \{0, 1, \ldots, 6, 10, 11, \ldots, 15\}$.

For $n \equiv 8$ (16), then $\rho(n, n-6) = \rho(n)$. Take $\dim(W) = n$, $s = 6$, $t = 0$ and $U = V_1$. Then $\sigma(n, n, n-6) \geq \sigma(n-6, n, n-6) \geq \rho(n)$.

For $n \equiv 9$ (16), then $\rho(n, n-6) = \rho(n-1)$. Then $\sigma(n, n, n-6) \geq \sigma(n-1, n-1, n-6) \geq \rho(n-1)$ by part (i).

For $n \equiv 7$ (16), then $\rho(n, n-6) = \rho(n+1)$. Take $\dim(W) = n+1$, $s = 1$, $t = 6$. We can take U to be V_1 providing $\theta(n+1, 1) + 6 \leq n+1$, that is if $\rho(n+1) \leq n-5$, which is true for $n \geq 8$. Then $\sigma(n, n, n-6) \geq \sigma(n, n-5, n-6) \geq \rho(n+1)$.

(iii) $\rho(n, n-7) = 8 \iff n = 16p + 8$. We get $h - g \equiv 8$ (16), with $g, h \in \{0, 1, \ldots, 7\}$, and the congruence cannot be satisfied.

For $n \equiv 8$ (16), then $\rho(n, n-7) = \rho(n)$. Take $\dim(W) = n$, $s = 7$, $t = 0$ and $U = V_1$. Then $\sigma(n, n, n-7) \geq \sigma(n-7, n, n-7) \geq \rho(n)$.

(iv) $\rho(n, n-5) = 9 \Rightarrow n$ is of the form $32p + 16 + j$ $(-5 \leq j \leq 5)$. So assume $\sigma(32p + 16 + j, 32p + 16 + j, 32p + 11 + j) \geq 10$. The exact sequence on RP^9 is

$$0 \longrightarrow G^5 \longrightarrow (32p + 16 + j)\epsilon \longrightarrow (32p + 16 + j)\lambda \longrightarrow H^5 \longrightarrow 0$$

which gives the congruence

$$h - g \equiv 16 + j \ (32) \qquad (-5 \leq j \leq 5).$$

Using the tables of [LR1], we see that the only elements of geometric dimension ≤ 5 in $\widetilde{KO}(RP^9)$ are $\{0, x, 2x, \ldots, 5x, 17x, 20x\}$. Now $16 + j \in \{11, 12, \ldots, 21\}$. For $j = \pm 5$ we get a contradiction.

For $n \equiv 21$ (32), $\rho(n, n-5) = \rho(n-5)$, so $\sigma(n, n, n-5) \geq \sigma(n-5, n-5, n-5) \geq \rho(n-5)$.

For $n \equiv 11$ (32) then $\rho(n, n-5) = \rho(n+5)$. Take $\dim(W) = n+5$ and $s = t = 5$. We can take $U = V_1$ providing $\theta(n+5, 5) + 5 \leq n+5$, that is if $\theta(n+5, 5) \leq n$. Write $n + 5 = 16 + 32m$ with $m \geq 1$, so we must show that $\theta(16 + 32m, 5) \leq 11 + 32m$. We have $\theta(16+32m, 5) \leq \theta(8+16m, 5) + 5(\rho(16+32m) - \rho(8+16m)) \leq 8 + 16m + 5(9-8) = 13 + 16m < 11 + 32m$ for all $m \geq 1$. This completes the proof of theorem 3.3.2.

For $k \geq 8$, our methods of estimating the dimension from above using Stiefel-Whitney classes or K-theory yield no results, since the corresponding congruences have solutions. When we consider the situation for $k = 9$, it turns out that we can construct spaces of dimension larger than $\rho(n, n-9)$.

Proposition 3.3.3 *There are infinitely many n for which we can construct spaces of $n \times n$ matrices of rank $n-9$ whose dimension exceeds $\rho(n, n-9)$.*

Proof By proposition 2.4.1, for any $k \leq n$ then

$$\sigma(n, n, n-k) \geq \sigma(n, n-k, n-k) \geq n-(n-k)+1 = k+1.$$

For $k = 9$, this gives $\sigma(n, n, n-9) \geq 10$. But if n is of the form $32p + 16$ then $\rho(n, n-9) = \rho(32p+16, 32p+7) = 9$.

We have seen that the upper bounds obtained using K-theory to study spaces of large rank $n \times n$ matrices are related to the Radon-Hurwitz function, which in turn is related to $v_2(n)$, the highest power of 2 dividing n. The final result of this chapter further illustrates the dependence of $\sigma(n, n, k)$ on $v_2(n)$ for large k.

Proposition 3.3.4 $\sigma(n, n, k) \leq 2^{v_2(n)}$ *for all* $k \geq n - 2^{v_2(n)} + 1$.

Proof Write $n = (2a+1)b$, where $b = 2^{v_2(n)}$, and suppose that $\sigma(n, n, k) \geq b+1$ for some fixed $k \geq n-b+1$. Then there exists an exact sequence

$$0 \longrightarrow G^{n-k} \longrightarrow n\epsilon \xrightarrow{\Phi} n\lambda \longrightarrow H^{n-k} \longrightarrow 0$$

over RP^b and isomorphisms $G^{n-k} \oplus F^k \cong n\epsilon$ and $H^{n-k} \oplus F^k \cong n\lambda$, where F^k denotes the k-dimensional bundle $\mathrm{Im}(\Phi)$. If $f(x), g(x)$ and $h(x)$ represent the total Stiefel-Whitney classes of the bundles F^k, G^{n-k} and H^{n-k} in $H^*(RP^b; Z_2)$, then

$$f(x)g(x) = 1 \quad \text{and} \quad f(x)h(x) = w(n\lambda) \bmod x^{b+1}.$$

Now b is the largest power of 2 dividing n, so $w(n\lambda) = 1 + x^b \bmod x^{b+1}$. Also, $g(x)$ and $h(x)$ are polynomials of degrees not exceeding $n-k$. For each $i \leq n-k$, express g_i and h_i in terms of f_j, $j \leq i$. We have $k \geq n-b+1$, so $n-k \leq b-1$ and $g_i = h_i$ for all $i \leq n-k$. Thus $g(x) = h(x)$. This gives the contradiction

$$1 = f(x)g(x) = f(x)h(x) = 1 + x^b \bmod x^{b+1}.$$

If $v_2(n) = 0$ then $b = 1$ and the proposition provides a more elementary proof (in the sense that no K-theory is required) of the fact that $\sigma(2p+1, 2p+1, 2p+1) = 1$. For $v_2(n) = 1$ then for $k = 4p+1$ or $4p+2$, $\sigma(4p+2, 4p+2, k) \leq 2 = \rho(4p+2, 4p+1)$. The upper bounds given by the next few cases are:

(i) $\sigma(8p+4, 8p+4, k) \leq 4$ for $8p+1 \leq k \leq 8p+4$.
(ii) $\sigma(16p+8, 16p+8, k) \leq 8$ for $16p+1 \leq k \leq 16p+8$.
(iii) $\sigma(32p+16, 32p+16, k) \leq 16$ for $32p+1 \leq k \leq 32p+16$.
(iv) $\sigma(64p+32, 64p+32, k) \leq 32$ for $64p+1 \leq k \leq 64p+32$.

Except for (i) and (ii), corresponding to $v_2(n) = 2$ and 3 respectively, the upper bounds may not be best possible. However, they do extend the range of values of k upon which we can place useful upper bounds.

Chapter 4

Low-rank Spaces

In this chapter we extend the results on $\sigma(m,n,k)$ by considering some spaces of real matrices where the rank is relatively small.

The first two sections are devoted to calculating the largest dimensions of all spaces of rank 3 or 4. In section 3 partial results on spaces of rank $k \le 9$ are given; to state the result for spaces of rank 9, the techniques of Stiefel-Whitney classes are combined with a construction (due to K.Y. Lam) of a nonsingular bilinear map $R^9 \times R^{16} \longrightarrow R^{16}$. Finally, we summarize the calculations of some low-dimensional cases with a table of $\sigma(n,n,k)$ for all $k \le n \le 12$.

Recall the results of theorem 2.4.3 (due to Rees [R2]):

(i) $\sigma(m,n,1) = \max(m,n)$.

(ii) $\sigma(n,n,2) = \begin{cases} n & n \text{ even.} \\ n-1 & n \text{ odd, } n > 3; \ \sigma(3,3,2) = 3. \end{cases}$

We begin by generalizing (ii) to spaces of arbitrary $m \times n$ rank 2 matrices.

Proposition 4.0.1 *Suppose $m \ge n \ge 2$. Then*

$$\sigma(m,n,2) = \begin{cases} m & m \textit{ even.} \\ m-1 & m \textit{ odd, } n > 3; \ \sigma(3,3,2) = 3, \ \sigma(3,2,2) = 2. \end{cases}$$

Proof We may assume $m \ge n$ since $\sigma(m,n,k) = \sigma(n,m,k)$ by proposition 2.4.1. Also, $\sigma(m,n,2) \le \max(m,n) = m$ by theorem 2.4.2.

There are two main cases to consider.

Case $m = 2p$. This follows from [R2]: every non-zero matrix of the form

$$\begin{bmatrix} a_1 & -b_1 & a_2 & -b_2 & \dots & a_p & -b_p \\ b_1 & a_1 & b_2 & a_2 & \dots & b_p & a_p \end{bmatrix}$$

has rank 2 and so $\sigma(2p, 2, 2) \geq 2p$ and thus $\sigma(2p, n, 2) \geq \sigma(2p, 2, 2) \geq 2p$.

Case $m = 2p+1$. Suppose $\sigma(2p+1, n, 2) \geq 2p+1$. Then there exists an exact sequence

$$0 \longrightarrow G^{2p-1} \longrightarrow (2p+1)\epsilon \xrightarrow{\Phi} n\lambda \longrightarrow H^{n-2} \longrightarrow 0$$

on RP^{2p}. Define F^2 to be the 2-dimensional bundle $\mathrm{Im}\Phi$. We have the isomorphisms

$$G^{2p-1} \oplus F^2 \cong (2p+1)\epsilon$$
$$H^{n-2} \oplus F^2 \cong n\lambda.$$

Assume $p \geq 2$. Over RP^d, with $d \geq 3$, any rank 2 real bundle must be isomorphic to one of 2ϵ, $\epsilon \oplus \lambda$ or 2λ (see [A2]). Hence $w(F^2) \in \{1, 1+x, 1+x^2\}$.

Suppose $w(F^2) = 1 + x^2$. The isomorphism $G^{2p-1} \oplus F^2 \cong (2p+1)\epsilon$ implies

$$w(G^{2p-1}) = w(F^2)^{-1} = (1+x^2)^{-1} = 1 + x^2 + x^4 + \ldots + x^{2p}.$$

In particular, $w_{2p}(G^{2p-1}) \neq 0$, a contradiction. Hence $w(F^2) = 1$ or $1 + x$. If n is even, i.e. $n = 2q$ ($q \leq p$), then we have $H^{2q-2} \oplus F^2 \cong 2q\lambda$. Now $w(F^2) = 1$ gives the contradiction $w_{2q}(H^{2q-2}) \neq 0$ and $w(F^2) = 1 + x$ gives $w(H^{2q-2}) = w((2q-1)\lambda)$ and so $w_{2q-1}(H^{2q-2}) \neq 0$. If $n = 2q + 1$ ($q \leq p$) then $H^{2q-1} \oplus F^2 \cong (2q+1)\lambda$ over RP^{2p}. Both $w(F^2) = 1$ and $w(F^2) = 1 + x$ give $w_{2q}(H^{2q-1}) \neq 0$.

The only case not covered above is that of spaces of 3×2 rank 2 matrices. We have $3 \geq \sigma(3, 2, 2) \geq \sigma(2, 2, 2) = 2$. In fact there is no 3-dimensional space, for such a space would lead to the exact sequence

$$0 \longrightarrow G^1 \longrightarrow 3\epsilon \xrightarrow{\Phi} 2\lambda \longrightarrow 0$$

over RP^2, which splits to give $G^1 \oplus 2\lambda \cong 3\epsilon$. But every one-dimensional bundle over RP^2 is isomorphic to ϵ or λ. Neither 2λ nor 3λ are stably trivial over RP^2 (though 4λ is), so we have a contradiction.

4.1 Rank 3 spaces

Theorem 4.1.1 *For $m \geq n \geq 3$,*

$$\sigma(m,n,3) = \begin{cases} m & m \equiv 0 \ (4). \\ m-1 & m \equiv 1 \ (4). \\ m-2 & m \equiv 2 \textit{ or } 3 \ (4). \end{cases}$$

Proof The low-dimensional spaces are dealt with first. From [R2], $\sigma(3,3,3) = 1$ and $\sigma(4,4,3) = 4$. Also, $4 \geq \sigma(4,3,3) \geq \rho(4) = 4$ and so $\sigma(4,3,3) = 4$.

Proposition 4.1.2 $\sigma(5,n,3) = 4$ $(3 \leq n \leq 5)$; $\sigma(6,n,3) = 4$ $(3 \leq n \leq 6)$.

Proof Because of the inequalities $\sigma(6,6,3) \geq \ldots \geq \sigma(6,3,3) \geq \ldots \sigma(4,3,3) = 4$ and $\sigma(6,6,3) \geq \sigma(5,5,3) \geq \ldots \geq \sigma(5,3,3) \geq \sigma(4,3,3) = 4$, it will be enough to show that $\sigma(6,6,3) \leq 4$. So suppose that $\sigma(6,6,3) \geq 5$. Then there exists

$$0 \longrightarrow G^3 \longrightarrow 6\epsilon \xrightarrow{\Phi} 6\lambda \longrightarrow H^3 \longrightarrow 0$$

on RP^4 and isomorphisms $G^3 \oplus F^3 \cong 6\epsilon$ and $H^3 \oplus F^3 \cong 6\lambda$. Recall that $\widetilde{KO}(RP^4)$ has order 8, so in the notation of the last chapter (where fx, gx, hx represent the images in K-theory of the bundles F^3, G^3 and H^3 and f, g and h are integers between 0 and 7), we have congruences $g + f \equiv 0$ (8) and $h + f \equiv 6$ (8). Since $w_4(r\lambda) \neq 0$ for $4 \leq r \leq 7$, we must have $f, g, h \in \{0, 1, 2, 3\}$. The only solution of the first congruence is $f = g = 0$. But then the second congruence implies $h = 6$, a contradiction.

Proposition 4.1.3 $\sigma(7,n,3) = 5$ $(3 \leq n \leq 7)$.

Proof We have $\sigma(7,7,3) \geq \sigma(7,6,3) \geq \ldots \geq \sigma(7,3,3) \geq 7-3+1 = 5$, so it will suffice to prove that $\sigma(7,7,3) \leq 5$. So suppose for a contradiction that $\sigma(7,7,3) \geq 6$. Then there exists an exact sequence

$$0 \longrightarrow G^4 \longrightarrow 7\epsilon \xrightarrow{\Phi} 7\lambda \longrightarrow H^4 \longrightarrow 0$$

on RP^5 and so $G^4 \oplus F^3 \cong 7\epsilon$ and $H^4 \oplus F^3 \cong 7\lambda$. The order of $\widetilde{KO}(RP^5)$ is again 8. As for the last example, the 3-dimensional bundle F^3 must have image $0, x, 2x$ or $3x$ in $\widetilde{KO}(RP^5)$. Also, $w_5(5\lambda) \neq 0$ and $w_5(7\lambda) \neq 0$, so $g, h \in \{0,1,2,3,4,6\}$. The congruence $g + f \equiv 0$ (8) has solutions $(f,g) = (0,0)$ and $(2,6)$. If $f = 0$ then $f + h \equiv 7$ (8) gives $h = 7$, and if $f = 2$ then $h = 5$. Both solutions for h are contradictions.

Consider now the general case. The following lemma will be used to prove the existence of certain spaces of matrices.

Lemma 4.1.4 *For $m_1, m_2 \geq k$, $\sigma(m_1+m_2,k,k) \geq \sigma(m_1,k,k)+\sigma(m_2,k,k)$.*

Proof of lemma Let M_1 and M_2 respectively be $m_1 \times k$ and $m_2 \times k$ matrices of maximal rank and representing vector spaces of maximal dimensions (i.e. $\sigma(m_1,k,k)$ and $\sigma(m_2,k,k)$). The $(m_1+m_2)\times k$ matrix formed by putting M_1 'next to' M_2 clearly has rank k and represents a vector space of dimension $\sigma(m_1,k,k)+\sigma(m_2,k,k)$.

The lemma can be rewritten in an obvious way:

Proposition 4.1.5 $\sigma(m,k,k) \geq max\{\sigma(i,k,k)+\sigma(j,k,k) \mid i+j=m;\ i,j \geq k\}$.

Write $m = 4p+q$ for some $p \geq 2$ and $0 \leq q \leq 3$. By the lemma,

$$\sigma(4p+q,n,3) \geq \sigma(4p,3,3) \geq p\sigma(4,3,3) = 4p.$$

For $q=3$ we get one extra dimension:

$$\sigma(4p+3,n,3) \geq \sigma(4p+3,3,3) \geq \sigma(4p,3,3)+\sigma(3,3,3) = 4p+\rho(3) = 4p+1.$$

Case $q=0$**.** The upper bound follows from $\sigma(m,n,k) \leq$max(m,n) (theorem 2.4.2).

Case $q=1$ **or** 2**.** Assume for a contradiction that $\sigma(4p+q,n,3) \geq 4p+1$. The exact sequence over RP^{4p} is

$$0 \longrightarrow G^{4p+q-3} \longrightarrow (4p+q)\epsilon \xrightarrow{\Phi} n\lambda \longrightarrow H^{n-3} \longrightarrow 0.$$

Let F^3 be the bundle ImΦ. The splitting gives

$$G^{4p+q-3} \oplus F^3 \cong (4p+q)\epsilon$$
$$H^{n-3} \oplus F^3 \cong n\lambda.$$

For $p=2$, then $G^{5+q} \oplus F^3 \cong (8+q)\epsilon$ over RP^8 and $g+f \equiv 0$ (16). Now $w_4(r\lambda) \neq 0$ for $4 \leq r \leq 7$ and $w_8(r\lambda) \neq 0$ for $8 \leq r \leq 15$. Hence $f \in \{0,1,2,3\}$. For $f=1,2,3$ then $g=15,14,13$ respectively and $w_8(G^{5+q}) = w_8(g\lambda) \neq 0$; for $f=0$ then $w_n(H^{n-3}) = w_n(n\lambda) \neq 0$ for $n \leq 8$; $n=9$ gives $w_8(H^6) = w_8(9\lambda) \neq 0$ and finally $n=10$ gives $w_8(H^7) = w_8(10\lambda) \neq 0$.

The situation for $p = 3$ is slightly different. The isomorphism $G^{9+q} \oplus F^3 \cong (12+q)\epsilon$ over RP^{12} gives $g + f \equiv 0$ (128). Recall from the previous chapter that the only elements in $\widetilde{KO}(RP^{12})$ that can have geometric dimension ≤ 3 are $0, x, 2x, 3x, 65x$ and $66x$. Hence g must be one of $0, 62, 63, 125, 126, 127$. For $g \neq 0$ the class $w_{12}(G^{9+q}) \neq 0$. For $g = 0$ then $w_n(H^{n-3}) = w_n(n\lambda) \neq 0$ gives a contradiction for all $n \leq 12$. Finally, if $n = 13$ then $w_{12}(H^{10}) = w_{12}(13\lambda) \neq 0$, and if $n = 14$ then $w_{12}(H^{11}) = w_{12}(14\lambda) \neq 0$.

For the general case we will again make use of the fact that for $d \geq 13$ the only elements of $\widetilde{KO}(RP^d)$ of geometric dimension ≤ 3 are rx with $0 \leq r \leq 3$. Taking $p \geq 4$ above will satisfy this condition. Hence the possibilities for $w(F^3)$ in $H^*(RP^{4p}; Z_2)$ are limited to $1,\ 1+x,\ 1+x^2,\ 1+x+x^2+x^3$.

If $w(F^3) = 1$ then $w(H^{n-3}) = w(n\lambda) \bmod x^{4p+1}$, and so $w_n(H^{n-3}) \neq 0$ gives a contradiction for all $n \leq 4p$. For $n = 4p+1$ then $w(H^{4p-2}) = w((4p+1)\lambda)$ and $w_{4p}(H^{4p-2}) \neq 0$; for $n = 4p+2$, $w_{4p}(H^{4p-1}) = w_{4p}((4p+2)\lambda) \neq 0$.

If $w(F^3) = 1 + x$ then $w(G^{4p+q-3}) = w(F^3)^{-1} = (1+x)^{-1} = 1 + x + \ldots + x^{4p} \bmod x^{4p+1}$, and $w_{4p}(G^{4p+q-3}) \neq 0$ gives a contradiction.

If $w(F^3) = 1 + x^2$ then $w(H^{n-3}) = w((n-2)\lambda) \bmod x^{4p+1}$ and so $w_{n-2}(H^{n-3}) \neq 0$ gives a contradiction providing $n - 2 \leq 4p$, i.e. $n \leq 4p+2$ (which is automatically satisfied since we began by assuming $n \leq 4p+2$).

Finally, assume $w(F^3) = (1+x)^3$. We have $w(G^{4p+q-3}) = w(F^3)^{-1}$ and

$$\begin{aligned}(1+x)^{-3} &= (1+x)(1+x)^{-4}\\ &= (1+x)(1+x^4)^{-1}\\ &= (1+x)(1+x^4+x^8+\ldots)\\ &= 1+x+x^4+x^5+x^8+x^9+\ldots.\end{aligned}$$

In particular, $w_{4p}(G^{4p+q-3}) \neq 0$.

Case $q = 3$. Here we assume $\sigma(4p+3, n, 3) \geq 4p+2$. The exact sequence on RP^{4p+1} is given by

$$0 \longrightarrow G^{4p} \longrightarrow (4p+3)\epsilon \xrightarrow{\Phi} n\lambda \longrightarrow H^{n-3} \longrightarrow 0.$$

For $p = 2$, this gives $G^8 \oplus F^3 \cong 11\epsilon$ and $H^{n-3} \oplus F^3 \cong n\lambda$ $(3 \leq n \leq 11)$ over RP^9. In chapter 3 it was shown that the only elements of geometric dimension ≤ 3 in $\widetilde{KO}(RP^9) \cong Z_{32}$ are the elements rx with $0 \leq r \leq 3$. Now $f + g \equiv 0$ (32). If $f = 1$

then $g = 31$ and $w_9(G^8) \neq 0$. Similarly, $f = 3 \Rightarrow g = 29$ and $w_9(G^8) \neq 0$. For $f = 2$ then $w_{n-2}(H^{n-3}) = w_{n-2}((n-2)\lambda) \neq 0$ providing $n-2 \leq 9$, i.e. $n \leq 11$. Finally, $f = 0$ gives $w_n(H^{n-3}) = w_n(n\lambda) \neq 0$ providing $n \leq 9$; for $n = 10$, $w_8(H^7) = w_8(10\lambda) \neq 0$ and for $n = 11$, $w_9(H^8) = w_9(11\lambda) \neq 0$.

For $p \geq 3$, then over RP^{4p+1}, $G^{4p} \oplus F^3 \cong (4p+3)\epsilon$ and $H^{n-3} \oplus F^3 \cong n\lambda$. Now $4p+1 \geq 13$ and so lemma 3.1.9 applies, hence $f \in \{0,1,2,3\}$.

If $f = 0$ then $w_n(H^{n-3}) = w_n(n\lambda) \neq 0$ providing $n \leq 4p+1$; $n = 4p+2$ gives $w_{4p}(H^{4p-1}) = w_{4p}((4p+2)\lambda) \neq 0$; $n = 4p+3$ gives $w_{4p+1}(H^{4p}) = w_{4p+1}((4p+3)\lambda) \neq 0$.

If $f = 1$ then $w(G^{4p}) = w(F^3)^{-1} = 1 + x + x^2 + \ldots$ and $w_{4p+1}(G^{4p}) \neq 0$.

If $f = 2$ then $w_{n-2}(H^{n-3}) = w_{n-2}((n-2)\lambda) \neq 0$ providing $n-2 \leq 4p+1$, i.e. $n \leq 4p+3$.

If $f = 3$ then $w(G^{4p}) = w(F^3)^{-1} = 1 + x + x^4 + x^5 + \ldots$ and $w_{4p+1}(G^{4p}) \neq 0$.

4.2 Rank 4 spaces

Before stating the general result, we first consider some low-dimensional cases.

Proposition 4.2.1 $\sigma(7,4,4) = \sigma(6,4,4) = \sigma(5,4,4) = 4$.

Proof Since $\sigma(7,4,4) \geq \sigma(6,4,4) \geq \sigma(5,4,4) \geq \sigma(4,4,4) = 4$, it will be enough to show that $\sigma(7,4,4) \leq 4$. So suppose that $\sigma(7,4,4) \geq 5$. Then there exists an exact sequence

$$0 \longrightarrow G^3 \longrightarrow 7\epsilon \xrightarrow{\Phi} 4\lambda \longrightarrow 0$$

on RP^4, which splits to give $G^3 \oplus 4\lambda \cong 7\epsilon$ and the congruence $g + 4 \equiv 0$ (8). Now $w_4(r\lambda) \neq 0$ for $4 \leq r \leq 7$ so $g \in \{0,1,2,3\}$. But $g+4$ cannot be congruent to 0 mod 8.

Proposition 4.2.2 $\sigma(6,5,4) = 4$.

Proof $\sigma(6,5,4) \geq \sigma(5,5,4) = 4$. Suppose $\sigma(6,5,4) \geq 5$. Then there exists

$$0 \longrightarrow G^2 \longrightarrow 6\epsilon \xrightarrow{\Phi} 5\lambda \longrightarrow H^1 \longrightarrow 0$$

over RP^4, which gives $G^2 \oplus 5\lambda \cong H^1 \oplus 6\epsilon$ and thus $g + 5 \equiv h$ (8). Over RP^4 we must have $g \in \{0,1,2\}$ and $h \in \{0,1\}$, which gives a contradiction.

Proposition 4.2.3 $\sigma(7,5,4)=5$.

Proof Recall from chapter 2 the method used for constructing spaces of matrices: if V is a real vector space with negative definite inner product then denote its Clifford Algebra by $C(V)$ and the span of $1\subset C(V)$ and $V\subset C(V)$ by V_1. If the n-dimensional vector space W is a $C(V)$ module then $\dim(V_1)\leq\rho(n)$. Finally, if $X,Y\subset W$ are subspaces of dimensions s and t respectively, and if $U\subset V_1$ has dimension l, with $Y\perp UX$, then $\sigma(n-s,n-t,n-s-t)\geq l$.

Take $\dim(W)=8$ and $\dim(V_1)=8$. Both W and V_1 may be regarded as the Cayley numbers K. Let $X=R\subset K$ and take $U\subset V_1$ to be the 5-dimensional subspace spanned by $\{1,e_1,\ldots,e_4\}$, so $\dim(UX)=5$. Then choose $Y=(UX)^{\perp}$ so $\dim(Y)=3$ and $\sigma(8-1,8-3,8-4)=\sigma(7,5,4)\geq 5$.

The following matrix is an example of a 5-dimensional space:

$$\begin{bmatrix} 0 & 0 & a_1 & -a_2 & a_3 & -a_4 & 0 \\ a_2 & -a_1 & 0 & 0 & a_5 & 0 & -a_4 \\ a_1 & a_2 & 0 & 0 & 0 & -a_5 & a_3 \\ a_4 & -a_3 & -a_5 & 0 & 0 & 0 & a_2 \\ a_3 & a_4 & 0 & a_5 & 0 & 0 & -a_1 \end{bmatrix}$$

Now suppose there exists a 6-dimensional space. Then there exists

$$0\longrightarrow G^3\longrightarrow 7\epsilon\xrightarrow{\Phi}5\lambda\longrightarrow H^1\longrightarrow 0$$

over RP^5 and isomorphisms

$$G^3\oplus F^4\cong 7\epsilon,\qquad \text{and}\qquad H^1\oplus F^4\cong 5\lambda.$$

Define mod 2 polynomials $f(x)=1+f_1x+f_2x^2+f_3x^3+f_4x^4$, $g(x)=1+g_1x+g_2x^2+g_3x^3$ and $h(x)=1+h_1x$ respectively as the total Stiefel-Whitney classes of the bundles F^4, G^3 and H^1.

The second isomorphism of bundles gives $h(x)f(x)=w(5\lambda)=1+x+x^4+x^5$ in $H^*(RP^5;Z_2)$. That is,

$$(1+h_1x)(1+f_1x+f_2x^2+f_3x^3+f_4x^4)=1+x+x^4+x^5 \bmod x^6.$$

We will compare the coefficients of x^i for $1 \leq i \leq 5$, to give relations amongst the coefficients of $f(x)$ and $h(x)$. Note that if $a \in Z_2$ then $a^2 = a$ and $a(a+1) = 0$.

x^1: $\quad f_1 + h_1 = 1 \Rightarrow h_1 = f_1 + 1.$
x^2: $\quad f_2 + f_1 h_1 = 0 \Rightarrow f_2 = f_1 h_1 = f_1(f_1+1) = 0.$
x^3: $\quad f_3 + f_2 h_1 = 0 \Rightarrow f_3 = 0$ since $f_2 = 0$.
x^4: $\quad f_4 + f_3 h_1 = 1 \Rightarrow f_4 = 1$ since $f_3 = 0$.
x^5: $\quad f_4 h_1 = 1 \Rightarrow h_1 = 1$ since $f_4 = 1$.

But then $f_1 = h_1 + 1 = 0$. Hence $h(x) = 1 + x$ and $f(x) = 1 + x^4$. The first isomorphism then gives

$$(1 + g_1 x + g_2 x^2 + g_3 x^3)(1 + x^4) = 1 \bmod x^6$$

Comparing coefficients of x^i for $1 \leq i \leq 3$ gives $g_1 = g_2 = g_3 = 0$ and so $g(x) = 1$. But then the coefficients of x^4 yield the contradiction $0 = 1$.

Remark Notice that only Stiefel-Whitney classes were used to give the upper bound here, though it could also have been realized by the usual ad-hoc combination of K-theory and Stiefel-Whitney classes. However, the above approach has two advantages. Firstly, one can sometimes obtain stronger results than using K-theory. For example, if one is unsuccessful in obtaining a contradiction over RP^{8n+4}, then there is no hope by using K-theory alone of obtaining a contradiction over RP^{8n+5}, since $\widetilde{KO}(RP^{8n+4})$ and $\widetilde{KO}(RP^{8n+5})$ both have the same order. But the Stiefel-Whitney calculation considers the class w_{8n+5}, which, as in the above case, may be crucial. Secondly, this method, whereby successive coefficients are expressed in terms of previous ones to obtain a contradiction, though tedious by hand, is systematic and so is easily translated into a computer program.

The next example further illustrates this method.

Proposition 4.2.4 $\sigma(7,7,4) = \sigma(7,6,4) = 6$.

Proof We have $\sigma(7,7,4) \geq \sigma(7,6,4) \geq \sigma(6,6,4) = 6$ (by [R2]). So it will be enough to show that $\sigma(7,7,4) \leq 6$. Suppose then that $\sigma(7,7,4) \geq 7$. Then there exists

$$0 \longrightarrow G^3 \longrightarrow 7\epsilon \xrightarrow{\Phi} 7\lambda \longrightarrow H^3 \longrightarrow 0$$

over RP^6, and isomorphisms $G^3 \oplus F^4 \cong 7\epsilon$ and $H^3 \oplus F^4 \cong 7\lambda$. As for the previous example, we define polynomials $f(x)$, $g(x)$ and $h(x)$ respectively as $w(F^4)$, $w(G^3)$ and $w(H^3)$. Over RP^6, $w(7\lambda) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$.

The bundle isomorphisms lead to the following relations, which are evaluated mod x^7:

$$(1+g_1x+g_2x^2+g_3x^3)(1+f_1x+f_2x^2+f_3x^3+f_4x^4)=1$$
$$(1+h_1x+h_2x^2+h_3x^3)(1+f_1x+f_2x^2+f_3x^3+f_4x^4)=1+x+x^2+x^3+x^4+x^5+x^6.$$

Equate coefficients of x^i for $1 \le i \le 4$:

$$\begin{aligned}
x^1\colon\ & g_1+f_1=0 \Rightarrow g_1=f_1;\\
& h_1+f_1=1 \Rightarrow h_1=f_1+1.\\
x^2\colon\ & g_2+g_1f_1+f_2=0 \Rightarrow g_2=f_2+f_1^2=f_2+f_1;\\
& h_2+h_1f_1+f_2=1 \Rightarrow h_2=f_2+(f_1+1)f_1+1=f_2+1.\\
x^3\colon\ & g_3+g_2f_1+g_1f_2+f_3=0 \Rightarrow g_3=f_3+f_2f_1+f_1(f_2+f_1)=f_3+f_1;\\
& h_3+h_2f_1+h_1f_2+f_3=1 \Rightarrow h_3=f_3+f_2(f_1+1)+f_1(f_2+1)+1\\
& \qquad =f_3+f_2+f_1+1.\\
x^4\colon\ & g_3f_1+g_2f_2+g_1f_3+f_4=0 \Rightarrow f_4=f_3g_1+f_2g_2+f_1g_3\\
& \qquad =f_3f_1+f_2(f_2+f_1)+f_1(f_3+f_1)\\
& \qquad =f_2+f_1f_2+f_1;\\
& h_3f_1+h_2f_2+h_1f_3+f_4=1 \Rightarrow f_4=f_3(f_1+1)+f_2(f_2+1)+f_1h_3+1\\
& \qquad =f_3f_1+f_3+f_1(f_3+f_2+f_1+1)+1\\
& \qquad =f_3+f_1f_2+1
\end{aligned}$$

Equating the two expressions for f_4 gives $f_3=f_2+f_1+1$. For the coefficient of x^5,

$$\begin{aligned}
x^5\colon\ & f_4g_1+f_3g_2+f_2g_3=0\\
& \Rightarrow (f_2+f_1f_2+f_1)f_1+(f_2+f_1+1)(f_2+f_1)+f_2(f_3+f_1)=0\\
& \Rightarrow f_1+f_2(f_2+f_1+1)+f_2f_1=0\\
& \Rightarrow f_1=0.
\end{aligned}$$

Now $f_1=0 \Rightarrow f_3=f_2+1$, $f_4=f_2$ and $g_3=f_3=f_2+1$. Finally, consider the coefficient of x^6,

$$x^6:\ f_4g_2+f_3g_3=0 \Rightarrow f_2f_2+(f_2+1)(f_2+1)=0 \Rightarrow f_2+f_2+1=0 \Rightarrow 1=0.$$

Thus we have a contradiction and so $\sigma(7,7,4) \le 6$, which completes the proof.

Proposition 4.2.5 *For* $0 \le q \le 3$, $p \ge 2$ *and* $4p+q \ge n \ge 4$ *then* $\sigma(4p+q,n,4)=4p$.

Proof The lower bound is given by

$$\sigma(4p+q,n,4) \ge \sigma(4p,4,4) \ge p\sigma(4,4,4)=4p.$$

For $p = 2$ and $4 \leq n \leq 8 + q \leq 11$ assume $\sigma(8 + q, n, 4) \geq 9$, so over RP^8 we get

$$0 \longrightarrow G^{4+q} \longrightarrow (8 + q)\epsilon \xrightarrow{\Phi} n\lambda \longrightarrow H^{n-4} \longrightarrow 0.$$

This gives $g + f \equiv 0$ (16) with $f \in \{0, 1, 2, 3, 4\}$. Then $g \in \{0, 12, 13, 14, 15\}$. Now $w_8(G^{4+q}) = w_8(g\lambda) \neq 0$ unless $g = 0$. But $g = 0 \Rightarrow f = 0 \Rightarrow w_n(H^{n-4}) = w_n(n\lambda) \neq 0$ for $n \leq 8$; $n = 9$ gives $w_8(H^5) = w_8(9\lambda) \neq 0$, $n = 10$ gives $w_8(H^6) = w_8(10\lambda) \neq 0$ and finally $n = 11$ gives $w_8(H^7) = w_8(11\lambda) \neq 0$.

For $p = 3$ and $4 \leq n \leq 12 + q \leq 15$ assume $\sigma(12 + q, n, 4) \geq 13$ to get

$$0 \longrightarrow G^{8+q} \longrightarrow (12 + q)\epsilon \xrightarrow{\Phi} n\lambda \longrightarrow H^{n-4} \longrightarrow 0$$

over RP^{12}, and the congruence $g + f \equiv 0$ (128). In chapter 3 it was shown that only the elements $0, x, 2x, 3x, 4x, 65x, 66x, 67x$ in $\widetilde{KO}(RP^{12})$ can be of geometric dimension ≤ 4. Hence $g \in \{0, 61, 62, 63, 124, 125, 126, 127\}$. But for $g \neq 0$ then $w_{12}(G^{8+q}) \neq 0$ and $g = 0$ gives $w_n(H^{n-4}) \neq 0$ for $n \leq 12$ and $w_{12}(H^{n-4}) \neq 0$ for $13 \leq n \leq 15$.

Now suppose that $p \geq 4$. For $q = 0$, $\sigma(4p, n, 4) \leq \max(4p, n) = 4p$. So for $1 \leq q \leq 3$ and $4 \leq n \leq 4p + q$, assume that $\sigma(4p + q, n, 4) \geq 4p + 1$. Then there exists an exact sequence

$$0 \longrightarrow G^{4(p-1)+q} \longrightarrow (4p + q)\epsilon \xrightarrow{\Phi} n\lambda \longrightarrow H^{n-4} \longrightarrow 0$$

over RP^{4p}. If F^4 is the 4-dimensional bundle $\mathrm{Im}\Phi$, then

$$G^{4(p-1)+q} \oplus F^4 \cong (4p + q)\epsilon, \quad \text{and} \quad H^{n-4} \oplus F^4 \cong n\lambda.$$

We showed in chapter 3 that the only elements of $\widetilde{KO}(RP^{4p})$ of geometric dimension ≤ 4 are the elements rx with $0 \leq r \leq 4$ and possibly $(2 + 2^{\phi(4p)-1})x$. So again the choices for $w(F^4)$ in $H^*(RP^{4p}; Z_2)$ are limited.

If $w(F^4) = 1$ then $w_n(H^{n-4}) = w_n(n\lambda) \neq 0$ for $n \leq 4p$. For $n = 4p + 1$, $w(H^{4p-3}) = (1 + x)^{4p+1} \bmod x^{4p+1}$ and $w_{4p}(H^{4p-3}) \neq 0$. For $n = 4p + 2$, $w_{4p}(H^{4p-2}) \neq 0$ and for $n = 4p + 3$, $w_{4p}(H^{4p-1}) \neq 0$.

If $w(F^4) = 1 + x$ then $w(G^{4(p-1)+q}) = (1 + x)^{-1} = 1 + x + x^2 + \ldots + x^{4p} \bmod x^{4p+1}$ and since $4(p - 1) + q < 4p$ for $1 \leq q \leq 3$ then $w_{4p}(G^{4(p-1)+q}) \neq 0$.

If $w(F^4) = 1 + x^2$ then $w(G^{4(p-1)+q}) = 1 + x^2 + x^4 + \ldots + x^{4p} \bmod x^{4p+1}$, so again we have $w_{4p}(G^{4(p-1)+q}) \neq 0$.

If $w(F^4) = (1+x)^3$ then $w(H^{n-4}) = w((n-3)\lambda)$ and so $w_{n-3}(H^{n-4}) \neq 0$ for $n-3 \leq 4p$, i.e $n \leq 4p+3$.

If $w(F^4) = (1+x)^4$ then $w(G^{4(p-1)+q}) = (1+x)^{-4} = 1 + x^4 + x^8 + \ldots + x^{4p} \bmod x^{4p+1}$ and $w_{4p}(G^{4(p-1)+q}) \neq 0$.

For p even, the element $(2 + 2^{\phi(4p)-1})x$ cannot have geometric dimension ≤ 4 (see proposition 3.2.8). So suppose $p = 2r+1$ and consider the element $(2 + 2^{\phi(8r+4)-1})x$ in $\widetilde{KO}(RP^{8r+4})$. Now $\phi(8r+4) = 4r+3$, so this element is really $(2 + 2^{4r+2})x$. It is easily dealt with: suppose that $w(F^4) = w((2 + 2^{4r+2})\lambda)$. We have

$$\begin{aligned} w((2 + 2^{4r+2})\lambda) &= (1+x)^2(1+x)^{2^{4r+2}} \\ &= (1+x^2)(1+x^{2^{4r+2}}). \end{aligned}$$

Now $x^{8r+5} = 0$ in $H^*(RP^{8r+4}; Z_2)$, and for any positive integer r, $2^{4r+2} > 8r+5$. So $w(F^4) = w(2\lambda)$, which has already been considered.

4.3 Further calculations

Proposition 4.3.1 *For* $0 \leq i \leq 4$,

$$\sigma(8p+q, 5+i, 5) = \begin{cases} 8p & 0 \leq q \leq 4,\ p \geq 1. \\ 8p+1 & q = 5,\ p \geq 1. \\ 8p+2 & q = 6,\ p \geq 1;\ \sigma(6,5,5) = 2. \\ 8p+3 & q = 7,\ p \geq 1;\ \sigma(7,5,5) = 3,\ \sigma(7,6,5) = 6. \end{cases}$$

Proof The main cases are considered separately.

Case $q = 0$. For $p \geq 1$ then $8p \geq \sigma(8p, 5+i, 5,) \geq \sigma(8p, 5, 5) \geq p\sigma(8, 5, 5) \geq 8p$.

Case $1 \leq q \leq 4$. Suppose $\sigma(5+i, 8p+q, 5)$ $(= \sigma(8p+q, 5+i, 5)) \geq 8p+1$. Then over RP^{8p} there exists an exact sequence

$$0 \longrightarrow G^i \longrightarrow (5+i)\epsilon \xrightarrow{\Phi} (8p+q)\lambda \longrightarrow H^{8p+q-5} \longrightarrow 0$$

For $i = 0$ then $(8p+q)\lambda \cong H^{8p+q-5} \oplus 5\epsilon$. Since H^{8p+q-5} has dimension $8p+q-5 \leq 8p-1$ for $1 \leq q \leq 4$ then we have the contradiction $w_{8p}(H^{8p+q-5}) = w_{8p}((8p+q)\lambda) \neq 0$.

For $i = 1$ then $G^1 \oplus (8p+q)\lambda \cong H^{8p+q-5} \oplus 6\epsilon$. Over RP^{8p}, G^1 must be isomorphic to ϵ or λ. If $G^1 \cong \epsilon$ then $w((8p+q)\lambda) = w(H^{8p+q-5})$ again; if $G^1 \cong \lambda$ then the isomorphism $G^1 \oplus F^5 \cong 6\epsilon$ gives $w(F^5) = w(\lambda)^{-1} = 1 + x + x^2 + \dots$ and we have the contradiction $w_6(F^5) \neq 0$.

For $i = 2$ then $G^2 \oplus (8p+q)\lambda \cong H^{8p+q-5} \oplus 7\epsilon$, with G^2 isomorphic to one of 2ϵ, $\epsilon \oplus \lambda$ or 2λ. The first two choices just give (at the cohomology level) the same contradictions as above. If $G^2 \cong 2\lambda$ then $G^2 \oplus F^5 \cong 7\epsilon$ gives $w(F^5) = w(2\lambda)^{-1} = 1 + x^2 + x^4 + \dots$ and so $w_6(F^5) \neq 0$.

The situation for $i = 3$ is similar. We have $w(G^3) = w(r\lambda)$ for some $0 \leq r \leq 3$. The only new case is $r = 3$: $w(F^5) = w(G^3)^{-1} = w(3\lambda)^{-1} = 1 + x + x^4 + x^5 + x^8 + x^9 + \dots$ and $w_8(F^5) \neq 0$.

For $i = 4$ then over RP^{8p} we have $w(G^4) = w(r\lambda)$ for some $0 \leq r \leq 4$. If $r = 4$ then $w(F^5) = w(4\lambda)^{-1} = 1 + x^4 + x^8 + \dots$ and $w_8(F^5) \neq 0$.

Case $q = 5$. The lower bound is given by $\sigma(8p+5, 5+i, 5) \geq \sigma(8p+5, 5, 5) \geq \sigma(8p, 5, 5) + \sigma(5, 5, 5) = 8p+1$. The calculations to prove the non-existence of the exact sequence

$$0 \longrightarrow G^i \longrightarrow (5+i)\epsilon \xrightarrow{\Phi} (8p+5)\lambda \longrightarrow H^{8p} \longrightarrow 0$$

over RP^{8p+1} are similar to those above. For $i = 4$ one also has to consider the possibility that $(2 + 2^{4p})\lambda$ has geometric dimension ≤ 4. But over RP^{8p+1}, this has the same Stiefel-Whitney classes as 2λ.

Case $q = 6$. The usual calculations with Stiefel-Whitney classes show that for $p \geq 1$ and $0 \leq i \leq 4$ then $\sigma(8p+6, 5+i, 5) \leq 8p+2$. The lower bound can be determined by $\sigma(8p+6, 5+i, 5) \geq \sigma(8p+6, 5, 5) \geq (8p+6)-5+1 = 8p+2$. Now $\sigma(6, 5, 5) \geq 6-5+1 = 2$. Hence $8p+2 = \sigma(8p+6, 5, 5) \geq \sigma(8p, 5, 5) + \sigma(6, 5, 5) \geq 8p + \sigma(6, 5, 5) \geq 8p+2$ and thus $\sigma(6, 5, 5) = 2$.

Case $q = 7$. The usual method gives $\sigma(8p+7, 5+i, 5) \leq 8p+3$. For the lower bound, $\sigma(8p+7, 5+i, 5) \geq \sigma(8p+7, 5, 5) \geq (8p+7)-5+1 = 8p+3$. Also, $\sigma(7, 5, 5) \geq 7-5+1 = 3$. Then $8p+3 = \sigma(8p+7, 5, 5) \geq \sigma(8p, 5, 5) + \sigma(7, 5, 5) \geq 8p+3$ and so $\sigma(7, 5, 5) = 3$. A construction to show that $\sigma(7, 6, 5) \geq 6$ is given in [R2]. Also, $\sigma(7, 6, 5) \leq \sigma(7, 7, 5) = 6$, so $\sigma(7, 6, 5) = 6$. This completes the proof.

Proposition 4.3.2 *For* $0 \le i \le 4$

$$\sigma(8p+q,6+i,6) = \begin{cases} 8p & 0 \le q \le 5,\ p \ge 1. \\ 8p+2 & 6 \le q \le 7,\ p \ge 1;\ \sigma(7,6,6) = 2. \end{cases}$$

Proof The non-existence of the exact sequence

$$0 \longrightarrow G^i \longrightarrow (6+i)\epsilon \xrightarrow{\Phi} (8p+q)\lambda \longrightarrow H^{8p+q-6} \longrightarrow 0$$

over RP^{8p} for $0 \le q \le 5$ and over RP^{8p+2} for $q = 6$ or 7 follows by the usual method. For $0 \le q \le 5$ the lower bound is given by $\sigma(8p+q,6+i,6) \ge \sigma(8p,6,6) \ge p\sigma(8,6,6) \ge 8p$. Also, $\sigma(8p+7,6+i,6) \ge \sigma(8p+6,6+i,6) \ge \sigma(8p,6,6) + \sigma(6,6,6) = 8p+2$, and $8p+2 = \sigma(8p+7,6,6) \ge \sigma(8p,6,6) + \sigma(7,6,6) \ge 8p + (7-6+1) = 8p+2$ which gives $\sigma(7,6,6) = 2$.

Similar results on the dimensions of spaces of $m \times n$ $(m \ge n)$ matrices of rank 7 or 8 can be obtained in much the same way. The maximum dimension follows the above pattern (i.e. it depends upon the residue class of m mod 8).

Proposition 4.3.3 *For* $0 \le i \le 4$,

(i) $\sigma(8p+q,7+i,7) = \begin{cases} 8p & p \ge 1,\ 0 \le q \le 6. \\ 8p+1 & p \ge 1,\ q = 7. \end{cases}$

(ii) $\sigma(8p+q,8+i,8) = 8p$ *(*$p = 1$ *and* $0 \le q \le 3$ *or* $p \ge 2$ *and* $0 \le q \le 7$*).*

Proof The details are omitted.

To obtain an equivalent result for spaces of rank 9, we invoke a construction of Lam [L1]. Recall from corollary 2.4.15 that we may interpret $\sigma(n,k,k)$ as the largest positive integer r for which there exists a nonsingular bilinear map $R^r \times R^k \longrightarrow R^n$.

Theorem 4.3.4 (Lam) *Denote the Cayley numbers by* K *and let* $u = (x_1,x_2), v = (y_1,y_2) \in K \times K$. *The following R-bilinear map is nonsingular.*

$$f : K^2 \times K^2 \to K^3; \qquad (u,v) \mapsto (x_1y_1 - \bar{y}_2x_2, y_2x_1 + x_2\bar{y}_1, x_2y_2 - y_2x_2).$$

In particular, restricting f to a suitable subspace of $K^2 \times K^2$ gives rise to a bilinear map $R^9 \times R^{16} \to R^{16}$. Hence $\sigma(16,9,9) \ge 16$ (and by theorem 2.4.2, $\sigma(16,9,9) = 16$).

Proposition 4.3.5

$$\sigma(16p+q,9,9) = \begin{cases} 16p & 0 \le q \le 8,\ p \ge 1. \\ 16p+q-8 & 9 \le q \le 15,\ p \ge 0. \end{cases}$$

Proof We supply the details for the lower bounds: For $0 \le q \le 8$ and $p \ge 1$ then $\sigma(16p+q,9,9) \ge \sigma(16p,9,9) \ge p\sigma(16,9,9) = 16p$. For $9 \le q \le 15$ and $p \ge 1$ then $\sigma(16p+q,9,9) \ge (16p+q)-9+1 = 16p+q-8$; also (given the upper bound by the usual method), $16p+q-8 = \sigma(16p+q,9,9) \ge \sigma(16p,9,9)+\sigma(q,9,9) \ge 16p+(q-9+1) = 16p+q-8$ and so $\sigma(q,9,9) = q-8$ for $9 \le q \le 15$.

For any k, we have the following.

Proposition 4.3.6 *For fixed* $0 \le i \le 3$ *and for all* $n \ge k+13$ *then*

$$\sigma(n,k+i,k) \le max_{(0 \le r \le i)}\ \{min\ (j \mid \tbinom{n+r}{j} \ne 0\ (2),\ n-k+1 \le j \le n)\}.$$

Proof Assume for some fixed $0 \le i \le 3$ that $\sigma(n,k+i,k) = d+1$. On RP^d we have

$$0 \longrightarrow G^i \longrightarrow (k+i)\epsilon \xrightarrow{\Phi} n\lambda \longrightarrow H^{n-k} \longrightarrow 0$$

and the isomorphism $G^i \oplus n\lambda \cong H^{n-k} \oplus (k+i)\epsilon$. Now for $n \ge k+13$ we have $\sigma(n,k+i,k) \ge \sigma(n,k,k) \ge n-k+1 \ge 14$. The bundle G^i has dimension $i \le 3$ and if $d \ge 13$ then the only elements which can have geometric dimension ≤ 3 in $\widetilde{KO}(RP^d)$ are rx with $0 \le r \le 3$. For fixed $r \le i$, if $w_j(H^{n-k}) = w_j(n+r)\lambda \ne 0$ for some $n-k+1 \le j \le n$ then we will get a contradiction. Hence we must take the minimum such j for each r and then the maximum of these integers will be an upper bound.

Discussion of the cases of spaces of $n \times n$ matrices of rank $n-3$ for $n < 8$ and of rank $n-4$ for $n < 12$ was deferred from the previous chapter. The results of this chapter have already filled in some of the gaps (e.g. $\sigma(6,6,3), \sigma(7,7,3), \sigma(7,7,4)$). The final result given here includes all the remaining cases of $\sigma(n,n,k)$ for $n \le 12$.

Proposition 4.3.7

(i) $\sigma(9,9,5) = 8$.
(ii) $\sigma(10,10,k) = 8\ (5 \le k \le 6)$.
(iii) $\sigma(11,11,k) = 8\ (5 \le k \le 7)$.
(iv) $\sigma(12,12,k) = 8\ (5 \le k \le 7)$.

Proof Define $S = \{(9,5),(10,5),(10,6),(11,5),(11,6),(11,7),(12,5),(12,6),(12,7)\}$. For each ordered pair $(n,k) \in S$, suppose $\sigma(n,n,k) \geq 9$. Then there exists an exact sequence

$$0 \longrightarrow G^{n-k} \longrightarrow n\epsilon \xrightarrow{\Phi} n\lambda \longrightarrow H^{n-k} \longrightarrow 0$$

over RP^8 and isomorphisms

$$G^{n-k} \oplus F^k \cong n\epsilon \text{ and } H^{n-k} \oplus F^k \cong n\lambda.$$

These give the congruences

$$g + f \equiv 0 \ (16) \text{ and } h + f \equiv n \ (16).$$

Notice that for each $(n,k) \in S$ we have $5 \leq k \leq 7$ and $4 \leq n-k \leq 7$. So F^k, G^{n-k}, and H^{n-k} have dimension ≤ 7. Now $w_8(r\lambda) \neq 0$ for $8 \leq r \leq 15$ gives $f,g,h \in \{0,1,\ldots,7\}$. To satisfy the first congruence we need $f = g = 0$. But then $h = n \in \{9,10,11,12\}$, giving a contradiction.

A Table for $\sigma(n,n,k)$

	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10	k=11	k=12
n=1	1	*	*	*	*	*	*	*	*	*	*	*
n=2	2	2	*	*	*	*	*	*	*	*	*	*
n=3	3	3	1	*	*	*	*	*	*	*	*	*
n=4	4	4	4	4	*	*	*	*	*	*	*	*
n=5	5	4	4	4	1	*	*	*	*	*	*	*
n=6	6	6	4	6	2	2	*	*	*	*	*	*
n=7	7	6	5	6	6	7	1	*	*	*	*	*
n=8	8	8	8	8	8	8	8	8	*	*	*	*
n=9	9	8	8	8	8	8	8	8	1	*	*	*
n=10	10	10	8	8	8	8	8	8	2	2	*	*
n=11	11	10	9	8	8	8	8	8	4	4	1	*
n=12	12	12	12	12	8	8	8	9	4	4	4	4

Chapter 5

Spaces of Real Symmetric Matrices

In this chapter we study the problem of determining the largest possible dimensions of certain linear spaces of fixed-rank symmetric matrices. The main result, proved in section 1, is that any space of odd rank must be 1-dimensional. A few spaces of even rank are considered in section 2.

Definition 5.0.1 *Let $\sigma_S(n,n,k)$ be the maximum dimension of a linear space of $n \times n$ real symmetric matrices all of whose non-zero entries have rank k.*

5.1 Spaces of odd rank

Theorem 5.1.1 $\sigma_S(n,n,2k+1) = 1$.

The proof is a consequence of a general argument concerning the way eigenvalues vary over certain linear spaces of matrices. Recall that all the eigenvalues of a real symmetric matrix are real.

Definition 5.1.2 *Let $n_+(A), n_-(A)$, and $n_0(A)$ respectively be the number of positive, negative and zero eigenvalues of a real $n \times n$ symmetric matrix A of rank k.*

We have $n_0(A) = n - k$ and $n_+(A) + n_-(A) = k$.

Denote by $M_S(n,R)$ the space of all real symmetric $n \times n$ matrices; let $V \subset M_S(n,R)$ be a linear space of dimension at least 2, satisfying $A \in V \backslash \{0\} \Rightarrow$ rank$(A)= k$.

Lemma 5.1.3 *$n_+(A)$ and $n_-(A)$ are constant on $V \backslash \{0\}$.*

Proof Let the eigenvalues of a matrix $A \in M_S(n, R)$ be $\lambda_1, \ldots, \lambda_n$ and write the characteristic polynomial as $c_t(A) = c_0 + c_1 t + \ldots + c_{n-1}t^{n-1} + t^n$. Let $R'[t]$ denote the polynomials in $R[t]$ whose roots are all real (so $c(M_S(n, R)) \subset R'[t]$). Consider the composition

$$\begin{aligned} \psi : \; M_S(n, R) &\longrightarrow R'[t] \longrightarrow R^n; \\ A &\mapsto c_t(A) \mapsto (\lambda_1, \ldots, \lambda_n). \end{aligned}$$

Here the eigenvalues are given their natural ordering as real numbers: $\lambda_i \leq \lambda_j$ for all $i \leq j$. (The ordering is unimportant in what follows though.) Observe that ψ is continuous: the coefficients of $c_t(A)$ are polynomials in the entries of A and so depend continuously on A. Moreover, the complex roots of any polynomial over C depend continuously on its coefficients (see [HJ]) and in particular, therefore, the (real) eigenvalues of A depend continuously on $c_t(A)$.

Let $\lambda_{i_1}, \ldots, \lambda_{i_k}$ be the non-zero eigenvalues of $A \in V\backslash\{0\}$. For each $j \leq k$, there is a neighbourhood of A in $V\backslash\{0\}$ in which λ_{i_j} has constant sign. The intersection of these neighbourhoods is a neighbourhood of A in which all the λ_{i_j} have constant sign. Hence n_+ and n_- are locally constant on $V\backslash\{0\}$. To prove that these functions are constant on all of $V\backslash\{0\}$, we use another lemma.

Lemma 5.1.4 *A locally constant function on a connected set is constant.*

Proof Suppose $f : X \longrightarrow R$ is a locally constant function on a connected set X and let $\alpha \in \mathrm{Im} f$. Then $f^{-1}(\alpha)$ and $\bigcup_{\beta \neq \alpha} f^{-1}(\beta)$ are open sets. But then $f^{-1}(\alpha) = X \backslash \bigcup_{\beta \neq \alpha} f^{-1}(\beta)$ is both open and closed in R^{n^2} and since X is connected, $\bigcup_{\beta \neq \alpha} f^{-1}(\beta)$ must be empty, giving $f^{-1}(\alpha) = X$.

Since V is of dimension at least 2, $V\backslash\{0\}$ is connected. Hence n_+ and n_- are constant on $V\backslash\{0\}$, which proves lemma 5.1.3.

Corollary 5.1.5 *If* $dim(V) \geq 2$ *then* $n_+(A) = n_-(A) = k/2$ *for all* $A \in V\backslash\{0\}$.

Proof Let $A \in V\backslash\{0\}$ have eigenvalues $\lambda_1, \ldots, \lambda_n$. There is a path in V from A to $-A$ avoiding the origin. The eigenvalues of $-A$ are $-\lambda_1, \ldots, -\lambda_n$. Hence $n_+(-A) = n_-(A)$ and $n_-(-A) = n_+(A)$. By lemma 5.1.3 $n_+(A) = n_+(-A) = n_-(A)$. Hence $n_+(A) = n_-(A) = k/2$.

Proof of theorem. Suppose V is a space of $n \times n$ symmetric matrices of rank $2k+1$ with $\dim(V) \geq 2$. Then for each $A \in V\backslash\{0\}$, $n_+(A) = n_-(A) = (2k+1)/2$. But $n_+(A)$ and $n_-(A)$ must be integers, so we have a contradiction.

5.2 Spaces of even rank.

Proposition 5.2.1 $\sigma(n,n,k) \geq \sigma_S(n,n,k) \geq \sigma_S(n-1,n-1,k)$.

Proof. The upper bound on $\sigma_S(n,n,k)$ is obvious. For the lower bound, consider a symmetric $n-1 \times n-1$ matrix A of rank k. Put A in the top left hand corner of an $n \times n$ matrix in which the remaining entries are zero. The matrix so formed is clearly symmetric and has the same rank as A.

Proposition 5.2.2 $\sigma_S(n,n,2k) \geq \sigma(n-k,k,k) \geq n-2k+1$.

Proof. Consider the space V represented by the $n \times n$ block matrix

$$\begin{bmatrix} 0_k & A \\ A^t & 0_{n-k} \end{bmatrix}$$

where 0_k is the zero matrix of order k and the $k \times (n-k)$ matrix A represents a space of dimension $\sigma(n-k,k,k)$. Then every non-zero matrix in V is symmetric and of rank $2k$. If A is the band matrix

$$\begin{bmatrix} a_1 & a_2 & \dots & a_{n-2k+1} & 0 & 0 & \dots & 0 \\ 0 & a_1 & a_2 & \dots & a_{n-2k+1} & 0 & \dots & 0 \\ \vdots & & \ddots & & & \ddots & & \vdots \\ 0 & 0 & \dots & \dots & a_1 & a_2 & \dots & a_{n-2k+1} \end{bmatrix}$$

then V has dimension $n-2k+1$.

5.2.1 The rank 2 case.

A 2 dimensional space of rank 2 symmetric 2×2 matrices is given by

$$\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix}$$

and so $\sigma_S(2,2,2) = 2$.

Proposition 5.2.3 $\sigma_S(2n+1,2n+1,2) = 2n$.

Proof. The lower bound comes from the previous construction; also, for $n > 1$ we have $\sigma_S(2n+1,2n+1,2) \leq \sigma(2n+1,2n+1,2) = 2n$ by theorem 2.4.3.

It remains to show that $\sigma_S(3,3,2) = 2$. Suppose V is a 3-dimensional space of 3×3 symmetric rank 2 matrices. Then corollary 5.1.5 applies, so for each $A \in V\backslash\{0\}$, $n_+(A) = n_-(A) = n_0(A) = 1$; moreover, there exists a nonsingular matrix P such that

$$PAP^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} =: A'.$$

If J is the (nonsingular) matrix

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then J and A' satisfy

$$JA'J^t = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =: A_1.$$

It is enough to consider spaces containing A_1: if V is a space not containing A_1, and if $B_1 \in V\backslash\{0\}$, then choose a nonsingular P such that $PB_1P^t = A_1$ and define

$$V_P := \{PBP^t \mid B \in V\}.$$

The vector space V_P contains A_1, consists entirely of symmetric matrices of rank 2 and is of the same dimension as V. Now consider general 3×3 symmetric matrices

$$A_2 = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, A_3 = \begin{bmatrix} p & q & r \\ q & s & t \\ r & t & u \end{bmatrix}.$$

We require every linear combination of A_1, A_2 and A_3 to have rank 2. The condition

$$(*) \qquad \det(xA_1 + yA_2 + zA_3) = 0 \qquad (\forall\ x, y, z \in R)$$

ensures that the rank of $xA_1 + yA_2 + zA_3$ is at most 2. Much information can be obtained by considering the two dimensional space spanned by A_1, A_2. On putting $z = 0$, $(*)$ can be written

$$(adf + 2bce - ae^2 - b^2f - c^2d)y^3 + (2ce - 2bf)xy^2 + (-f)x^2y = 0 \ (\forall x, y \in R).$$

The only way the determinant can vanish for every choice of x and y is if each of the coefficients vanish:

$$\begin{array}{ll} x^2y: & -f = 0 \Rightarrow f = 0. \\ xy^2: & 2ce - 2bf = 0 \Rightarrow ce = 0. \\ y^3: & adf + 2bce - ae^2 - b^2f - c^2d = 0 \Rightarrow ae^2 + dc^2 = 0. \end{array}$$

There are 3 possibilities:

$e = c = 0 \Rightarrow a, b, d$ arbitrary.
$e \neq 0 \Rightarrow a = c = 0.$
$c \neq 0 \Rightarrow d = e = 0.$

So A_2 can be of the following three general types:

$$\begin{bmatrix} a & b & 0 \\ b & d & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b & 0 \\ b & d & e \\ 0 & e & 0 \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix}$$

Call these types 1,2 and 3 respectively. Of course, we could have put $y = 0$ in $(*)$ and deduced the same possibilities for A_3.

To eliminate the various cases we either find some x, y and z such that a (non-trivial) linear combination $xA_1 + yA_2 + zA_3$ has rank 1 or exhibit a linear dependence amongst the A_i.

Case: A_2 of type 1. If A_3 is also of type 1 then we are essentially just looking at a space of 2×2 matrices, which has dimension at most 2. If A_3 is of type 2, consider the linear combination $L_1 = xA_1 + yA_2 + zA_3$ given by

$$L_1 = \begin{bmatrix} ya & x + yb + zq & 0 \\ x + yb + zq & yd + zs & zt \\ 0 & zt & 0 \end{bmatrix}.$$

The coefficient of yz^2 in $(*)$ is $-at^2$; $t \neq 0$ so $a = 0$. Choose $z = 0$ and $x = -yb$ to get

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & yd & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which has rank 1 unless $d = 0$. But $d = 0$ implies $A_2 = bA_1$ and we have established a linear dependence.

Now let A_3 be of type 3. An arbitrary linear combination of A_1, A_2 and A_3 is

$$L_2 = \begin{bmatrix} ya+zp & x+yb+zq & zr \\ x+yb+zq & yd & 0 \\ zr & 0 & 0 \end{bmatrix}.$$

Interchanging the first two rows and columns in L_2 gives the same form as L_1.

Case: A_2 of type 2. Suppose A_3 is also of type 2. Then

$$L_3 = \begin{bmatrix} 0 & x+yb+zq & 0 \\ x+yb+zq & yd+zs & ye+zt \\ 0 & ye+zt & 0 \end{bmatrix}$$

Put $y = t,\ z = -e$ and $x = eq - tb$ to get

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & td-es & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This has rank 1 unless $td = es$, that is if $s = (t/e)d$. But then

$$\tfrac{t}{e}A_2 + (q - \tfrac{tb}{e})A_1 = \begin{bmatrix} 0 & \frac{tb}{e} & 0 \\ \frac{tb}{e} & \frac{td}{e} & t \\ 0 & t & 0 \end{bmatrix} + \begin{bmatrix} 0 & q-\frac{tb}{e} & 0 \\ q-\frac{tb}{e} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & q & 0 \\ q & s & t \\ 0 & t & 0 \end{bmatrix} = A_3$$

If A_3 has type 3 then

$$L_4 = \begin{bmatrix} zp & x+yb+zq & zr \\ x+yb+zq & yd & ye \\ zr & ye & 0 \end{bmatrix}.$$

The coefficient of xyz in the determinant of L_4 is $2re$, which is non-zero since both r and e are non-zero. Hence this case cannot arise.

Case: A_2 of type 3. The only remaining possibility is for A_2 and A_3 to both be of type 3. This gives

$$L_5 = \begin{bmatrix} ya+zp & x+yb+zq & yc+zr \\ x+yb+zq & 0 & 0 \\ yc+zr & 0 & 0 \end{bmatrix}$$

Interchange the first two rows and columns to get the same form for L_5 as L_3.

It is conjectured that $\sigma_S(n, n, 2) = n - 1$ for all $n > 2$, though we have been unable to obtain a general proof for the cases where n is even. An outline of the proof for $n = 4$ is given below.

Proposition 5.2.4 $\sigma_S(4, 4, 2) = 3$.

Proof As for the previous example, corollary 5.1.5 applies. One then considers the 2-dimensional space spanned by A_1 and an arbitrary 4×4 symmetric matrix A_2, where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_5 & a_6 & a_7 \\ a_3 & a_6 & a_8 & a_9 \\ a_4 & a_7 & a_9 & a_{10} \end{bmatrix}.$$

For every x and y, the determinant of $xA_1 + yA_2$ must vanish, as must the coefficient of t in the characteristic polynomial $c_t(xA_1 + yA_2)$.

These conditions imply that A_2 must be of the form

$$A_2 = \begin{bmatrix} a_1 & a_2 & a_3 & pa_3 \\ a_2 & a_5 & a_3 & pa_3 \\ a_3 & a_3 & 0 & 0 \\ pa_3 & pa_3 & 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} a_1 & a_2 & a_3 & pa_3 \\ a_2 & a_5 & a_3 & -pa_3 \\ a_3 & a_3 & 0 & 0 \\ pa_3 & -pa_3 & 0 & 0 \end{bmatrix}.$$

Here p is fixed. For a 4 dimensional space to exist, one considers 8 cases (corresponding to the 2 choices for each of A_2, A_3 and A_4. It is then easy to establish that the space spanned by these matrices has dimension at most 3.

This method can in principle be adapted to higher dimensional spaces: A_1 is the $n \times n$ diagonal matrix with diagonal entries $1, -1, 0, \ldots, 0$. One must show that if A_2 is an arbitrary $n \times n$ symmetric matrix, then for $xA_1 + yA_2$ to have rank 2 for all $x, y \in R$, A_2 must be of the form

$$\begin{bmatrix} X & Y \\ Y^t & 0_{n-2} \end{bmatrix}$$

where X is an arbitrary 2×2 symmetric matrix and Y is of the form

$$\begin{bmatrix} c & p_1c & p_2c & \ldots & p_{n-3}c \\ c & \pm p_1c & \pm p_2c & \ldots & \pm p_{n-3}c \end{bmatrix}.$$

The number of cases to check grows rapidly.

5.2.2 Some large-rank examples.

A theorem of Adams, Lax and Phillips [ALP] - on the maximum dimensions of linear spaces of real, complex and quaternionic nonsingular matrices and their Hermitian counterparts - is discussed in chapter 1. In the notation of this thesis, the result concerning real symmetric matrices is

Theorem 5.2.5 (Adams, Lax, and Phillips) *For n even, $\sigma_S(n,n,n) = \rho(\frac{n}{2}) + 1$.*

Of course, if n is odd then theorem 5.1.1 applies and the maximum dimension is 1. Alternatively, $\sigma_S(2n+1, 2n+1, 2n+1) \leq \sigma(2n+1, 2n+1, 2n+1) = \rho(2n+1) = 1$.

Recall also from theorem 2.4.6 that $\sigma(n,n,n-1) = \rho(n,n-1)$, $n \neq 3,7$. In particular, the inequality $\sigma(n,n,n-1) \leq \rho(n,n-1)$ holds for all positive integers. Combining these with proposition 5.2.1 gives

$$\rho(n)+1 = \sigma_S(2n,2n,2n) \leq \sigma_S(2n+1,2n+1,2n) \leq \sigma(2n+1,2n+1,2n) \leq \rho(2n+1,2n).$$

By writing n in the form $n = (2a+1)2^{c+4d}$ for suitable a, c and d with $d \geq 1$, we can determine the cases for which the upper and lower bounds are close or coincide.

	$c=0$	$c=1$	$c=2$	$c=3$
$\rho(n)+1$	$2+8d$	$3+8d$	$5+8d$	$9+8d$
$\rho(2n+1,2n)$	$2+8d$	$4+8d$	$8+8d$	$9+8d$

This gives our final result:

Proposition 5.2.6 *For $d \geq 1$.*

(i) $n = (2a+1)2^{4d} \Rightarrow \sigma_S(2n+1, 2n+1, 2n) = 2+8d.$

(ii) $n = (2a+1)2^{1+4d} \Rightarrow 3+8d \leq \sigma_S(2n+1, 2n+1, 2n) \leq 4+8d.$

(iii) $n = (2a+1)2^{2+4d} \Rightarrow 5+8d \leq \sigma_S(2n+1, 2n+1, 2n) \leq 8+8d.$

(iv) $n = (2a+1)2^{3+4d} \Rightarrow \sigma_S(2n+1, 2n+1, 2n) = 9+8d.$

Appendix A

On the Geometric Dimension of $(4k+2)\lambda$ over RP^n

Notation In what follows $r\lambda$ denotes the r-fold Whitney sum of the tautological line bundle λ over RP^n. The symbol ϵ is used to represent both real and complex trivial line bundles over RP^n and their images in K-theory (the context makes this clear). The generators of $\widetilde{KO}(RP^n)$ and $\widetilde{KU}(RP^n)$ - of orders $2^{\phi(n)}$ and $2^{[n/2]}$ respectively - are denoted by x, y. The image of a real [complex] r-plane bundle η in $KO(RP^n)$ [$KU(RP^n)$] is written $\eta \sim mx + r\epsilon$ [$my + r\epsilon$], where m is an integer. Finally, recall that $\mathrm{Spin}^c(r)$ is defined as $\mathrm{Spin}(r) \times_{Z_2} \mathrm{Spin}(2)$ and is a double cover of $SO(r) \times SO(2)$.

Proposition If $(4k+2)\lambda$ has geometric dimension ≤ 4 on RP^n then $4k \equiv 0 \bmod 2^{[n/2]}$.

Proof Suppose that $(4k+2)\lambda$ has geometric dimension ≤ 4 over RP^n. Then there exists an $SO(4)$ bundle E on RP^n with $E \sim (4k+2)x + 4\epsilon$. Denote by E_C the complexification of E. Thus $E_C \sim (4k+2)y + 4\epsilon$. Finally, define $\widetilde{E}$ to be the $SO(6)$ bundle $E \oplus 2\lambda$ so $\widetilde{E} \sim (4k+4)x + 6\epsilon$. Since $w_2(\widetilde{E}) = 0$, the structural group lifts to $\mathrm{Spin}(6)$. Moreover, $\widetilde{E}$ comes from an $SO(4) \times SO(2)$ bundle, so the structural group pulls back to $\mathrm{Spin}^c(4)$ via the following pullback diagram

$$\begin{tikzcd} \mathrm{Spin}^c(4) \arrow[r] \arrow[d] & \mathrm{Spin}(6) \arrow[d] \\ SO(4) \times SO(2) \arrow[r] & SO(6). \end{tikzcd}$$

Now $\mathrm{Spin}(4) \cong S^3 \times S^3 \cong SU(2) \times SU(2)$ and $\mathrm{Spin}(2) \cong S^1$. Hence $\mathrm{Spin}^c(4)$ can be described as $(SU(2) \times SU(2) \times S^1)/Z_2$, where (A, B, z) is identified with $(-A, -B, -z)$. Also, $\mathrm{Spin}^c(4)$ is isomorphic to $S(U(2) \times U(2))$ (which consists of pairs (C,D) of 2×2 unitary matrices satisfying $\det(C).\det(D) = 1$) via the map $(A, B, z) \mapsto (zA, z^{-1}B)$.

Identify R^4 with the set

$$\left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \mid a, b \in C \right\}$$

and define the $SO(4)$ representation ϕ of $S(U(2) \times U(2))$ by the action on R^4 given by $X \mapsto CXD^T$ $(X \in R^4,\ (C, D) \in S(U(2) \times U(2)))$.

Lemma $\phi(\tilde{E}) \cong E$.

Proof E is an $SO(4)$ bundle. The action on R^4 of the (connected) double cover $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$ of $SO(4)$ is given by the map $X \mapsto AXB^T$. The $SO(4)$ bundle $\phi(\tilde{E})$ is the image under ϕ of the $\mathrm{Spin}^c(4)$ bundle $\tilde{E}$. Identify $\mathrm{Spin}^c(4)$ with $S(U(2) \times U(2))$ by the above isomorphism. The action on R^4 of the double cover $SU(2) \times SU(2) \times S^1$ of $\mathrm{Spin}^c(4)$ is given by $X \mapsto (zA)X(z^{-1}B)^T = AXB^T$. This induces an action of $\mathrm{Spin}^c(4)$ on R^4, given by the same formula.

Let $\phi_C = \phi \otimes C$ be the $U(4)$ representation corresponding to the complexification of ϕ, and define the $U(2)$ representations F^+, F^- of $S(U(2) \times U(2))$ by:

$$\begin{aligned} F^+ &: S(U(2) \times U(2)) \longrightarrow U(2); \quad (A, B) \mapsto A \\ F^- &: S(U(2) \times U(2)) \longrightarrow U(2); \quad (A, B) \mapsto B \end{aligned}$$

Lemma $\phi_C \cong F^+ \otimes F^-$.

Proof Since $S(U(2) \times U(2))$ is a compact connected Lie group, the action on R^4 under ϕ is determined by the action of its maximal torus, which can be described as the set of ordered pairs of matrices of the form

$$\left(\begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i(\theta_2 - \theta_1)} \end{bmatrix}, \begin{bmatrix} e^{i(\theta_3 - \theta_2)} & 0 \\ 0 & e^{-i\theta_3} \end{bmatrix} \right).$$

The action is given by

$$AXB^T = \begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i(\theta_2 - \theta_1)} \end{bmatrix} \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} e^{i(\theta_3 - \theta_2)} & 0 \\ 0 & e^{-i\theta_3} \end{bmatrix}$$

$$= \begin{bmatrix} ae^{i(\theta_1 + \theta_3 - \theta_2)} & be^{i(\theta_1 - \theta_3)} \\ -\bar{b}e^{i(\theta_3 - \theta_1)} & \bar{a}e^{i(\theta_2 - \theta_1 - \theta_3)} \end{bmatrix}.$$

Let $\alpha = \theta_1 + \theta_3 - \theta_2$, $\beta = \theta_1 - \theta_3$. Choosing a suitable basis, we can write the $U(4)$ matrix representing the action of the maximal torus for ϕ_C as

$$\begin{bmatrix} e^{i\alpha} & 0 & 0 & 0 \\ 0 & e^{-i\alpha} & 0 & 0 \\ 0 & 0 & e^{i\beta} & 0 \\ 0 & 0 & 0 & e^{-i\beta} \end{bmatrix}.$$

Let $\{e_1, e_2\}$ be the usual basis for C^2, so $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ is a basis for $C^2 \otimes C^2 \cong C^4$. The action of the maximal torus for $F^+ \otimes F^-$ is given by:

$$F^+ \otimes F^-(e_1 \otimes e_1) = F^+(e_1) \otimes F^-(e_1) = e^{i\theta_1} e_1 \otimes e^{i(\theta_3 - \theta_2)} e_1 = e^{i\alpha}(e_1 \otimes e_1).$$
$$F^+ \otimes F^-(e_1 \otimes e_2) = F^+(e_1) \otimes F^-(e_2) = e^{i\theta_1} e_1 \otimes e^{-i\theta_3} e_2 = e^{i\beta}(e_1 \otimes e_2).$$
$$F^+ \otimes F^-(e_2 \otimes e_1) = F^+(e_2) \otimes F^-(e_1) = e^{i(\theta_2 - \theta_1)} e_2 \otimes e^{i(\theta_3 - \theta_2)} e_1 = e^{-i\beta}(e_2 \otimes e_1).$$
$$F^+ \otimes F^-(e_2 \otimes e_2) = F^+(e_2) \otimes F^-(e_2) = e^{i(\theta_2 - \theta_1)} e_2 \otimes e^{-i\theta_3} e_2 = e^{-i\alpha}(e_2 \otimes e_2).$$

The actions of ϕ_C and $F^+ \otimes F^-$ agree on the maximal torus of $S(U(2) \times U(2))$ and so the representations are isomorphic, which proves the lemma.

Applying the lemma to $\widetilde{E}$ gives the isomorphism $\phi_C(\widetilde{E}) \cong F^+(\widetilde{E}) \otimes F^-(\widetilde{E})$. Now $\phi_C(\widetilde{E}) \sim (4k+2)y + 4\epsilon$. Write $F^+(\widetilde{E}) \sim f_+ y + 2\epsilon$ and $F^-(\widetilde{E}) \sim f_- y + 2\epsilon$. We get an equation in $KU(RP^n)$

$$(4k+2)y + 4\epsilon = (f_+ y + 2\epsilon)(f_- y + 2\epsilon)$$

which leads to the congruence

$$2k + 1 \equiv f_+ + f_- - f_+ f_- \quad \mod 2^{[n/2]-1}$$

Notice that the left hand side is odd, so f_+, f_- cannot both be even. The next calculation shows that f_+ and f_- have the same parity and because of the above, must both be odd.

The determinant may be interpreted as the top exterior power operation, and the condition $\det(C).\det(D) = 1$ for $(C, D) \in S(U(2) \times U(2))$ gives the following equation in $KU(RP^n)$

$$\lambda_2(F^+(\widetilde{E})).\lambda_2(F^-(\widetilde{E})) = \epsilon.$$

One can also show that $\lambda_2(my + 2\epsilon) = -m^2y + 2my + \epsilon$. So we have,

$$(-f_+^2 y + 2f_+ y + \epsilon)(-f_-^2 y + 2f_- y + \epsilon) = \epsilon$$

which leads to the congruence

$$2f_+ + 2f_- - f_+^2 - f_-^2 \equiv 2(2f_+ - f_+^2)(2f_- - f_-^2) \qquad \bmod 2^{[n/2]}.$$

Working mod 2 gives

$$f_+^2 \equiv f_-^2 \qquad \bmod 2$$

so f_+, f_- have the same parity and must be odd. In fact we can say slightly more:

Proposition $\lambda_2(F^+(\widetilde{E})) = \lambda_2(F^-(\widetilde{E})) = y + \epsilon$.

Proof $F^+(\widetilde{E}),\ F^-(\widetilde{E})$ are $U(2)$ bundles, so $\lambda_2(F^+(\widetilde{E})),\ \lambda_2(F^-(\widetilde{E}))$ are complex line bundles and so stably must be ϵ or $y + \epsilon$. Suppose that $\lambda_2(F^+(\widetilde{E})) \sim \epsilon$. Then

$$-f_+^2 y + 2f_+ y + \epsilon = \epsilon$$

and thus

$$f_+^2 \equiv 2f_+ \qquad \bmod 2^{[n/2]}.$$

which implies that f_+ is even: a contradiction. Hence $\lambda_2(F^+(\widetilde{E})) \sim y + \epsilon$. The calculation for $\lambda_2(F^-(\widetilde{E}))$ is identical.

To complete the proof we use (complex) γ operations on the classes in $KU(RP^n)$ of the $U(2)$ bundles $F^+(\widetilde{E}),\ F^-(\widetilde{E})$. The polynomial $(1 + yt)^{f_+}$ is of degree ≤ 2.

In particular,

$$\binom{f_+}{3} y^3 = \tfrac{f_+(f_+-1)(f_+-2)}{3.2.1} 4y = 0$$

which gives the congruence

$$f_+(f_+ - 1)(f_+ - 2) \equiv 0 \ (2^{[n/2]-1}).$$

Now f_+ and $f_+ - 2$ are odd, so these factors can be dropped. Hence

$$f_+ \equiv 1 \bmod 2^{[n/2]-1}.$$

The calculation for f_- is identical. Substituting for f_+ and f_- in the congruence

$$2k + 1 \equiv f_+ + f_- - f_+ f_- \ (2^{[n/2]-1})$$

and multiplying by 2 gives $4k \equiv 0 \ (2^{[n/2]})$, which completes the proof.

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